STRONG SUMS OF PROJECTIONS IN VON NEUMANN FACTORS.

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Abstract. This paper presents necessary and sufficient conditions for a positive bounded operator on a separable Hilbert space to be the sum of a finite or infinite collection of projections (not necessarily mutually orthogonal), with the sum converging in the strong operator topology if the collection is infinite. A similar necessary condition is given when the operator and the projections are taken in a type II von Neumann factor, and the condition is proven to be also sufficient if the operator is "diagonalizable". A simpler necessary and sufficient condition is given in the type III factor case.

1. Introduction

Which positive bounded operators on a separable Hilbert space can be written as sums of projections? For finite sums, Fillmore asked this question and obtained the characterizations of the finite rank operators that are sums of projections [6, Theorems 1] (see Corollary 2.5 below) and of the bounded operators that are sums of two projections [6, Theorems 2] (see Proposition 2.10 below.)

For infinite sums with convergence in the strong operator topology, this question arose naturally from work on ellipsoidal tight frames by Dykema, Freeman, Kornelson, Larson, Ordower, and Weber in [5]. They proved that a sufficient condition for a positive bounded operator $A \in B(H)^+$ to be the sum of projections is that its essential norm $||A||_e$ is larger than one ([5, Theorem 2]). This result served as a basis for further work by Kornelson and Larson [16] and then by Antezana, Massey, Ruiz, and Stojanoff [1] on the decomposition of positive operators into (strongly converging) sums of rank-one positive operators of preset norms.

The same question can be asked relative to a von Neumann algebra M. We say that an operator $A \in M^+$ is a strong sum of projections if there exists a collection of (not necessarily mutually orthogonal or commuting) projections $P_j \in M$ with cardinality $N \leq \infty$, for which $A = \sum_{j=1}^{N} P_j$ and the series converges in the strong operator topology if $N = \infty$. The main goal of this article is to answer the question of which operators are strong sums of projections.

To simplify the treatment, we consider only von Neumann factors, and we further assume that they are σ-finite (i.e., countably decomposable) so that all infinite projections are equivalent. Thus let H be a complex infinite dimensional Hilbert space and $M \subset B(H)$ be a σ-finite von Neumann factor. If M is of type I, we will identify it with $B(H)$ (hence we will assume that H is separable), and denote by Tr the usual normalized trace such that Tr $P = 1$ for any rank-one projection P. If M is of type II, τ will denote the faithful positive semifinite normal trace, unique up to scalar multiples in the type II_∞ case and normalized

Date: November 18, 2008.
1991 *Mathematics Subject Classification.* Primary: 47C15, Secondary: 46L10.
Key words and phrases. Sums of projections.

by $\tau(I) = 1$ in the type II_1 case. If M is only assumed to be semifinite, i.e., it is of type I or type II unless specified, we will generically denote its trace by τ.

The conditions for A to be a strong sum of projections are expressed in terms of the *excess* and the *defect* parts of A. Given $A \in M^+$, we denote by

χ_A	the spectral measure of A
$R_A = \chi_A(0, \|\|A\|\|]$	the range projection of A
$A_+ := (A - I)\chi_A(1, \|\|A\|\|]$	the excess part of A
$A_- := (I - A)\chi_A(0, 1)$	the defect part of A.

Thus we have the decomposition

$$A = A_+ - A_- + R_A. \tag{1}$$

A positive operator A is said to be *diagonalizable* if $A = \sum \gamma_j E_j$ for some $\gamma_j > 0$ and mutually orthogonal projections $\{E_j\}$ in M. Diagonalizable operators are also called *discrete* and are the most accessible operators in a type II factor (e.g., see [3].)

The main results of this article are collected in the following theorem.

Theorem 1.1. *Assume that M is a σ-finite von Neumann factor and $A \in M^+$.*

(i) *Let M be of type I. Then A is a strong sum of projections if and only if either $Tr\,(A_+) = \infty$ or $Tr\,(A_-) \leq Tr\,(A_+) < \infty$ and $Tr\,(A_+) - Tr\,(A_-) \in \mathbb{N} \cup \{0\}$.(Theorems 6.6, 4.3, and 3.3.)*

(ii) *Let M be of type II and A be diagonalizable. Then A is a strong sum of projections if and only if $\tau(A_+) \geq \tau(A_-)$. The condition is necessary even when A is not diagonalizable. (Theorems 6.6, 5.2, and 3.3.)*

(iii) *Let M be of type III. Then A is a strong sum of projections if and only if either $||A|| > 1$ or A is a projection. (Corollary 6.4 and Theorem 3.3.)*

Remark 1.2. *The statement (i) above extends the sufficient condition obtained in* [5, Theorem 2]. *In fact, it is elementary to show that $||A||_e > 1$ implies that $Tr\,(A_+) = \infty$; however, the reverse implication is false.*

The necessary conditions in Theorem 1.1 are obtained via the frame theory type construction of Proposition 3.1 that links decomposability of an operator A into a strong sum of projections to the condition that the identity is the "diagonal" of W^*AW for some partial isometry W with $W^*W = R_A$. For instance, the integrality condition in the $B(H)$ case (Theorem 1.1 (i)) when Tr $(A_+) < \infty$ emerges naturally from the fact that Tr $(A_+) -$ Tr (A_-) coincides with the trace of the projection $I - WW^*$.

A modification of these arguments provides an alternative proof of the necessity of the "integrality condition" for diagonals of projections in Kadison's [9, Theorem 15] that identifies explicitly the integer as the difference of traces of two projections (Corollary 3.6.)

The basic tool for all the sufficient conditions is provided by a 2×2 matrix construction that decomposes certain diagonal matrices into the sum of a projection and a rank-one "remainder" (Lemma 2.1). This lemma serves also several other purposes: when applied to finite matrices it provides in Corollary 2.5 another proof of Fillmore's characterization of finite sums of projections [6, Theorem 1]. It can be applied to (finite) sums of scalar multiples of mutually orthogonal equivalent projections in a C^*-algebra (Lemma 2.6). It

also provides in the von Neumann algebra setting a short constructive proof (Proposition 2.10) of Fillmore's characterization of sums of two projections [6, Theorem 2, Corollary].

As the results of [5] suggest, the most tractable case is the "infinite" one. The key special case (Lemma 6.1) is when A is an infinite sum of scalar multiples of mutually orthogonal equivalent projections in M and the sum of the coefficients in the corresponding expansion of A_+ diverges. Based on this lemma we obtain the sufficiency in Theorem 1.1 for part (iii), for part (i) when $\mathrm{Tr}\,(A_+) = \infty$, and for part (ii) when $\tau(A_+) = \infty$.

For the more delicate "finite trace" case in $B(H)$, i.e., when $\mathrm{Tr}\,(A_+) < \infty$, we diagonalize A_+ and A_- and then apply iteratively Lemma 2.1 , which provides canonically a sequence of projections. The strong convergence of the series of these projections is proven by reducing the problem to a finite dimensional construction and to three infinite dimensional special cases (Lemmas 2.3 , 4.1, 4.2, and Theorem 4.3.)

When M is of type II, Lemma 2.1 can also be applied to diagonalizable operators, where the strong convergence of the "remainders" is obtained by showing that they converge in the trace-norm (Lemma 5.1). Example 5.3 exhibits a non-diagonalizable operator that is the sum of two projections. It remains open whether the condition $\tau(A_+) \geq \tau(A_-)$ is always sufficient for A to be the strong sum of projections.

Von Neumann algebras are by no means the only setting in which positive operators may be decomposed into sums of projections. In a separate paper ([12]), we will investigate the same problem for positive operators in the multiplier algebra $M(\mathcal{A} \otimes K)$ where $\mathcal{A}$ is a $\sigma-$unital purely infinite simple C^*-algebra.

The first and second named authors were participants in the NSF supported Workshop in Linear Analysis and Probability, Texas A&M University, 2006, where they first heard from David Larson about the results in [5] and [16] that stimulated this project.

2. The matrix construction

We start with a simple lemma which will be used in our key constructions.

Lemma 2.1. *Let e and f be two orthogonal unit vectors in H. For every $\mu \geq 0$ and $0 \leq \lambda \leq 1$, let*

$$\nu := \begin{cases} \frac{(1-\lambda)\lambda}{(1+\mu-\lambda)(\mu+\lambda)} & \text{for } \mu \neq 0 \\ 1 & \text{for } \mu = 0 \end{cases} \quad \text{and} \quad \rho := \begin{cases} \frac{(1-\lambda)\mu}{\mu+\lambda} & \text{for } \mu \neq 0 \\ 0 & \text{for } \mu = 0 \end{cases} \tag{2}$$

and let

$$w := \sqrt{\rho} f - \sqrt{1-\rho}\, e \quad \text{and} \quad v := \sqrt{\nu} f + \sqrt{1-\nu}\, e. \tag{3}$$

Then $w \otimes w$ and $v \otimes v$ are rank-one projections and

$$(1+\mu)(e \otimes e) + (1-\lambda)(f \otimes f) = w \otimes w + (1+\mu-\lambda)(v \otimes v). \tag{4}$$

Proof. It is immediate to verify that $0 \leq \nu, \rho \leq 1$, w and v are unit vectors, and hence $w \otimes w$, $v \otimes v$ are rank-one projections with range contained in $\mathrm{span}\{e, f\}$. Their matrix representations with respect to the basis $\{e, f\}$ are, respectively,

$$\begin{pmatrix} 1-\rho & -\sqrt{\rho(1-\rho)} \\ -\sqrt{\rho(1-\rho)} & \rho \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1-\nu & \sqrt{\nu(1-\nu)} \\ \sqrt{\nu(1-\nu)} & \nu \end{pmatrix}.$$

An elementary computation shows that

$$\begin{pmatrix} 1+\mu & 0 \\ 0 & 1-\lambda \end{pmatrix} = \begin{pmatrix} 1-\rho & -\sqrt{\rho(1-\rho)} \\ -\sqrt{\rho(1-\rho)} & \rho \end{pmatrix} + (1+\mu-\lambda) \begin{pmatrix} 1-\nu & \sqrt{\nu(1-\nu)} \\ \sqrt{\nu(1-\nu)} & \nu \end{pmatrix}$$

and hence (4) holds. □

If we do not require the orthogonality of the vectors e and f, we still obtain the decomposition in (4), but the vectors w and v are no longer obtained as simply as in (3). With a slight generalization and a reformulation in terms of rank-one projections, we have

Lemma 2.2. *Let P, Q be rank-one projections in $B(H)$ and let $a \leq c \leq b$. Then there are projections $P' \sim Q' \sim P$ for which $aP + bQ = cP' + (a+b-c)Q'$.*

Proof. The cases when $P = Q$ or when $a = 0$ or $c = a$ or $c = b$ being trivial, we assume that $P \neq Q$ and that $0 < a < c < b$. Diagonalize the positive rank-two operator A. Then $A = a'E + b'F$ where E and F are two mutually orthogonal rank-one projections, $0 < a' \leq a < c < b \leq b'$, and $a' + b' = a + b$. Without loss of generality we can assume that $c = 1$, and now the conclusion follows from Lemma 2.1. □

A generalization of Lemma 2.2 provides the algorithm for constructing frame perturbations in [14].

The following lemma is obtained by iterative applications of Lemma 2.1 and serves several complementary purposes: it illustrates in the simpler finite-dimensional case a construction that is applicable also in the cases of infinite dimensions, it is a key ingredient in the proof of Theorem 4.3, and it provides another proof of Fillmore's characterization of finite sums of rank-one projections [6, Theorem 1].

Lemma 2.3. *Let $A \in B(H)^+$ be a finite rank operator and set*

$$A = \sum_{j=1}^{n}(1+\mu_j)(e_j \otimes e_j) + \sum_{i=1}^{m}(1-\lambda_i)(f_i \otimes f_i)$$

where $\{e_1, \dots, e_n, f_1, \dots, f_m\}$ is a collection of mutually orthogonal unit vectors, $\mu_j > 0$ and $0 \leq \lambda_i < 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. If all the eigenvalues of A are greater than 1, set $m = 0$, i.e., drop the sum involving the λ_i. Similarly, if all the eigenvalues of A are less than or equal to 1, set $n = 0$.

(i) *Assume that $k := Tr\,(A) - Tr\,(R_A) \in \mathbb{N} \cup \{0\}$. Then A is the sum of $n+m+k$ rank-one projections.*

(ii) *Assume that $0 \leq \sum_{j=1}^n \mu_j - \sum_{i=1}^m \lambda_i \leq \max\{\mu_j\}$. Then there are $n+m$ rank-one projections $P_1, P_2, \cdots, P_{m+n}$ for which*

$$A = \sum_{h=1}^{m+n-1} P_h + \Big(1 + \sum_{j=1}^{n}\mu_j - \sum_{i=1}^{m}\lambda_i\Big)P_{m+n}. \tag{5}$$

Proof. First notice that $A_+ = \sum_{j=1}^n \mu_j (e_j \otimes e_j)$ and $A_- = \sum_{i=1}^m \lambda_i (f_i \otimes f_i)$, hence by (1)

$$\text{Tr }(A) - \text{Tr }(R_A) = \text{Tr }(A_+) - \text{Tr }(A_-) = \sum_{j=1}^{n}\mu_j - \sum_{i=1}^{m}\lambda_i.$$

(i) To avoid triviality, assume that $A \neq 0$, and in particular that $n \neq 0$. If $k > 0$, let

$$A_1 := \mu_1(e_1 \otimes e_1) + \sum_{j=2}^{n}(1+\mu_j)(e_j \otimes e_j) + \sum_{i=1}^{m}(1-\lambda_i)(f_i \otimes f_i).$$

Then $A = e_1 \otimes e_1 + A_1$, $A_1 \geq 0$, $R_{A_1} = R_A$, and $\text{Tr }(A_1) = \text{Tr }(A) - 1$, whence $\text{Tr }(A_1) - \text{Tr }(R_{A_1}) = k - 1$. Iterating, we decompose A into the sum of k rank-one projections and a positive operator A_k with $\text{Tr }(A_k) = \text{Tr }(R_{A_k})$. Thus, we can simply assume that $k = 0$. Hence $\sum_{j=1}^{n} \mu_j = \sum_{i=1}^{m} \lambda_i$ and also $m \neq 0$.

Now we start by the decomposition

$$(1-\lambda_1)(f_1 \otimes f_1) + (1+\mu_1)(e_1 \otimes e_1) = P_1 + (1+\delta_1)(v_1 \otimes v_1)$$

where $\delta_1 := \mu_1 - \lambda_1$ and P_1 and $v_1 \otimes v_1$ are the rank-one projections prescribed by Lemma 2.1. Then either $\delta_1 = 0$ and $n = m = 1$, in which case A is the sum of two rank-one projections, or one of the three conditions hold: $\delta_1 > 0$, in which case $m > 1$; $\delta_1 < 0$, in which case $n > 1$; or $\delta_1 = 0$ and $(n, m) \neq (1, 1)$, in which case $n > 1$ and $m > 1$. Notice that v_1 is orthogonal to each e_j and f_i for $j > 1$ and $i > 1$ if any, so Lemma 2.1 yields again the decomposition

$$\begin{cases} (1+\delta_1)(v_1 \otimes v_1) + (1-\lambda_2)(f_2 \otimes f_2) & \text{if } \delta_1 > 0 \\ (1+\delta_1)(v_1 \otimes v_1) + (1+\mu_2)(e_2 \otimes e_2) & \text{if } \delta_1 < 0 \\ (1-\lambda_2)(f_2 \otimes f_2) + (1+\mu_2)(e_2 \otimes e_2) & \text{if } \delta_1 = 0 \end{cases} = P_2 + (1+\delta_2)(v_2 \otimes v_2)$$

where P_2 and $v_2 \otimes v_2$ are rank one projections and

$$\delta_2 = \begin{cases} \mu_1 - \lambda_1 - \lambda_2 & \text{if } \delta_1 > 0 \\ \mu_1 + \mu_2 - \lambda_1 & \text{if } \delta_1 < 0 \\ \mu_2 - \lambda_2 & \text{if } \delta_1 = 0 \end{cases}$$

In general after q steps, we have

$$A_q := \sum_{j=1}^{n'}(1+\mu_j)(e_j \otimes e_j) + \sum_{i=1}^{m'}(1-\lambda_i)(f_i \otimes f_i) = \sum_{j=1}^{q} P_j + (1+\delta_q)(v_q \otimes v_q) \tag{6}$$

where $\delta_q = \sum_{j=1}^{n'} \mu_j - \sum_{i=1}^{m'} \lambda_i$ and $n', m' \in \mathbb{N}$, $n' \leq n, m' \leq m$. We continue the process until we "run out" of summands to which apply Lemma 2.1. This occurs only when $n' = n$ and $m' = m$. Indeed, if $n' = n$ but $m' < m$, then $\delta_q = \sum_{j=1}^{n} \mu_j - \sum_{i=1}^{m'} \lambda_i = \sum_{i=m'+1}^{m} \lambda_i > 0$ and thus we can further decompose $(1+\delta_q)v_q \otimes v_q + (1-\lambda_{m'+1})(f_{m'+1} \otimes f_{m'+1})$ into the sum of a rank-one projection and a positive remainder. The case when $m' = m$ but $n' \neq n$ is similar. But when $n' = n$ and $m' = m$, then $\delta_q = 0$ and hence $A = A_q$ is the sum of $\text{Tr }(A) = n + m + k$ rank-one projections.

(ii) Assume without loss of generality that $\max\{\mu_j\}$ occurs for $j = n$, i.e., that

$$\sum_{j=1}^{n-1} \mu_j \leq \sum_{i=1}^{m} \lambda_i \leq \sum_{j=1}^{n} \mu_j.$$

We can carry on the same construction process as in (i). If after the q steps that lead to the decomposition (6) we have $n' = n$ and $m' \neq m$, then $\delta_q \geq \sum_{i=m'+1}^{m} \lambda_i \geq 0$ and we can

continue the process. If we have $m' = m$ but $n' \neq n$ then

$$\delta_q = \sum_{j=1}^{n'} \mu_j - \sum_{i=1}^{m} \lambda_i \leq \sum_{j=1}^{n-1} \mu_j - \sum_{i=1}^{m} \lambda_1 \leq 0.$$

and in this case too we can continue the process. Thus the process terminates only when $n' = n$ and $m' = m$ and thus (5) holds.

□

Remark 2.4. *The condition in (ii) is necessary, because if* (5) *holds, then*

$$1 + \sum_{j=1}^{n} \mu_j - \sum_{i=1}^{m} \lambda_i \leq ||A|| = 1 + \max\{\mu_j\}.$$

This lemma provides a constructive proof of Fillmore's characterization of finite sums of finite projections [6, Theorem 1] that does not depend on the mean value theorem (see also [5, Proposition 6]).

Corollary 2.5. [6, Theorem 1] *Let $A \in B(H)^+$ be a finite rank operator. Then A is the sum of projections if and only if Tr $A \in \mathbb{N}$ and Tr $A \geq$ Tr (R_A).*

Proof. The sufficiency is given by Lemma 2.3 (i). For the necessity, assume that $A = \sum_{j=1}^{k} P_i$ is a sum of projections and by further decomposing them if necessary, assume that they all have rank one. Then Tr $A = k \in \mathbb{N}$ and, clearly, rank $A \leq k$. .

□

The matrix construction in Lemma 2.1 extends to C^*-algebras and hence in particular to von Neumann algebras. It is well known that given a collection $\{E_j\}_{j=1}^{n}$ of mutually orthogonal equivalent projections in a C^*-algebra $\mathcal{A}$, we can chose a corresponding set of matrix units and hence an embedding of $M_n(\mathbb{C})$ into $\mathcal{A}$. Thus by Lemma 2.1 and Corollary 2.5 we obtain:

Lemma 2.6. *Let $\mathcal{A}$ be a C^*-algebra.*

(i) *If E and F are two mutually orthogonal equivalent projections in $\mathcal{A}$, $0 \leq \lambda \leq 1$, and $\mu \geq 0$, then there are two projections P_- and P_+ in $\mathcal{A}$, with $P_- \sim P_+ \sim E$, for which $(1+\mu)E + (1-\lambda)F = P_- + (1+\mu-\lambda)P_+$.*

(ii) *If $A = \sum_{j=1}^{n} \gamma_j E_j$ for some mutually orthogonal equivalent projections $E_j \in \mathcal{A}$ and some scalars $\gamma_j > 0$ with $\sum_{j=1}^{n} \gamma_j = k \in \mathbb{N}$ and $k \geq n$, then A is the sum of k equivalent projections in $\mathcal{A}$.*

Remark 2.7.

(i) *The embedding of $M_n(\mathbb{C})$ into $\mathcal{A}$ depends not only on the projections E_j but also on the matrix units. However, once these matrix units are chosen, the construction in Lemma 2.1 assigns the decomposition in a canonical way.*

Explicitly for the $n = 2$ case, let $V \in \mathcal{A}$ be a partial isometry with $E = V^*V$ and $F = VV^*$, then the projections P_- and P_+ obtained from this embedding and the formulas in Lemma 2.1 are

$$\begin{aligned} P_- &:= (1-\rho)E - \sqrt{\rho(1-\rho)}(V+V^*) + \rho F \\ P_+ &:= (1-\nu)E + \sqrt{\nu(1-\nu)}(V+V^*) + \nu F. \end{aligned} \tag{7}$$

It can also be verified directly that setting $W := \sqrt{1-\rho}E - \sqrt{\rho}V$*, we get* $W^*W = E$ *and* $WW^* = P_-$*. Thus* W *is a partial isometry and hence* P_- *is a projection and* $P_- \sim E$*. Similarly,* $P_+ \sim E$*.*

(ii) *More generally, if* $0 \le a \le c \le b$*, set*

$$\rho := \begin{cases} \frac{a(b-c)}{c(b-a)} & \text{if } b \neq c \\ 0 & \text{if } b = c \end{cases} \quad \text{and} \quad \nu := \begin{cases} \frac{a(c-a)}{(a+b-c)(b-a)} & \text{if } b \neq c \\ 1 & \text{if } b = c \end{cases}.$$

Then with P_- *and* P_+ *as in (7),* $bE + aF = cP_- + (a+b-c)P_+$*.*

Lemma 2.8. *Let* P*,* Q *be finite equivalent commuting projections in a von Neumann algebra* M *and let* $0 \le a < b$ *and* $a \le c \le b$*. Then there are projections* $P' \sim Q' \sim P$ *in* M *for which* $aP + bQ = cP' + (a+b-c)Q'$*.*

Proof. By the assumption of finiteness, we have the cancellation $P - PQ \sim Q - PQ$. By Lemma 2.6 (see also Remark 2.7 (ii)), there are projections $P' \sim Q' \sim P - PQ$ in M, with $P' \vee Q' \le P - PQ + Q - PQ$, for which $a(P-PQ) + b(Q-PQ) = cP' + (a+b-c)Q'$. But then,

$$aP + bQ = a(P-PQ) + b(Q-PQ) + (a+b)PQ = c(P'+PQ) + (a+b-c)(Q'+PQ).$$

Since $P' \perp PQ$ and $Q' \perp PQ$, $P'+PQ$ and $Q'+PQ$ are projections and both are equivalent to P.

□

Remark 2.9.

(i) *If the projections* P *and* Q *are not finite, cancellation might fail and indeed the property itself might fail. For instance if* P *is infinite but* $P \neq I$*, then there are no projections* P' *and* Q' *for which* $\frac{1}{5}P + I = \frac{2}{5}P' + \frac{4}{5}Q'$*. Indeed, otherwise* $I - Q' = \frac{2}{5}P' - \frac{1}{5}P - \frac{1}{5}Q'$*, whence* $||I - Q'|| \le \frac{4}{5}$ *and hence* $Q' = I$*. But then,* $\frac{1}{5}P + \frac{1}{5}I = \frac{2}{5}P'$*, whence* $P = P' = I$*, against the assumption.*

(ii) *Lemma 2.8 holds also for every* C^**-algebra* $\mathcal{A}$ *with the cancellation property (e.g., AF-algebras.)*

The proof of Lemma 2.1 can be used also to obtain a simple constructive proof of Fillmore's [6, Theorem 2, Corollary] characteriziation of the operators in $B(H)$ that are sums of two projections. The same characterization holds for von Neumann algebras.

Proposition 2.10. *Let* M *be a von Neumann algebra and* $A \in M$ *with* $0 \le A \le 2I$*. Then* A *is the sum of two projections in* M *if and only* $A = E \oplus B$ *where* E *is a (possibly zero) projection in* M *and there is a unitary* $U \in M$ *that commutes with* E *and for which* $UBU^* = 2R_B - B$*.*

Proof. To prove the sufficiency, it is obviously enough to consider the case when $E = 0$ and $R_A = I$, i.e., $UAU^* = 2I - A$. Let $\mathcal{A}$ be the (abelian) von Neumann algebra generated by A. Since $UAU^* = 2I - A \in \mathcal{A}$, it follows that $U\mathcal{A}U^* \subset \mathcal{A}$ and hence $U^2\mathcal{A}(U^2)^* \subset U\mathcal{A}U^*$. Since

$$A = 2I - UAU^* = 2I - U(2I - UAU^*)U^* = U^2A(U^2)^*,$$

it follows that $\mathcal{A} = U^2\mathcal{A}(U^2)^*$. Thus $\mathcal{A} = U\mathcal{A}U^*$, i.e., $U \cdot U^*$ is a conjugation of $\mathcal{A}$. In particular, for every Borel set $\Omega \subset [0,2]$ there is a Borel set $\Omega_U \subset [0,2]$ for which $U\chi_A(\Omega)U^* = \chi_A(\Omega_U)$.

Let $E_t := \chi_A[0,t) \in \mathcal{A}$ for $t \in [0,2]$ be the spectral resolution of A. Then

$$A = \int_{[0,1)} t dE_t + \chi_A\{1\} + \int_{(1,2]} t dE_t = U(2I - A)U^*$$
$$= U\Big(\int_{(1,2]} (2-t) dE_t\Big)U^* + U\chi_A\{1\}U^* + U\Big(\int_{[0,1)} (2-t) dE_t\Big)U^*.$$

It is now clear that

$$U\chi_A\{1\}U^* = \chi_A\{1\}, \quad U\chi_A[0,1)U^* = \chi_A(1,2], \quad \text{and} \quad U\chi_A(1,2]U^* = \chi_A[0,1).$$

In particular, $\int_{(1,2]} t dE_t = U\Big(\int_{[0,1)}(2-t)dE_t\Big)U^*$. Thus

$$A = \int_{[0,1)} t dE_t + \chi_A\{1\} + U\big(\int_{[0,1)} (2-t) dE_t\big)U^*.$$

Let

$$2P_- := \int_{[0,1)} t dE_t - U\int_{[0,1)} \sqrt{t(2-t)} dE_t - \int_{[0,1)} \sqrt{t(2-t)} dE_t U^* + U\int_{[0,1)} (2-t) dE_t U^*$$
$$2P_+ := \int_{[0,1)} t dE_t + U\int_{[0,1)} \sqrt{t(2-t)} dE_t + \int_{[0,1)} \sqrt{t(2-t)} dE_t U^* + U\int_{[0,1)} (2-t) dE_t U^*.$$

Then both P_- and P_+ belong to M and are selfadjoint. Since $\chi_A[0,1) \perp U\chi_A[0,1)U^*$, it is simple to verify that P_-, P_+ are idempotents and hence are projections. Furthermore $P_- + P_+ = A - \chi_A\{1\}$, hence $P_- \perp \chi_A\{1\}$ and thus $P_- + \chi_A\{1\}$ is also a projection, which completes the proof of the sufficiency. The necessity follows as in Fillmore's proof in [6, Theorem 2, Corollary] from the analysis of the relative position of two projections which holds for general von Neumann algebras (e.g., see [19, Pgs 306-308]), and hence, applies without changes to our setting. ☐

Remark 2.11. *With the notations of the above proof, if $\mathcal{A}$ is a masa, then it cannot be singular, since U belongs to the normalizer $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ but does not belong to $\mathcal{A}$, as otherwise $A = I$, against the assumption that $\mathcal{A}$ is a masa.*

3. The necessary condition

Proposition 3.1. *Let $A \in M^+$ and let $N \in \mathbb{N} \cup \{\infty\}$. Then the following conditions are equivalent.*

(i) *There is a partial isometry V with $V^*V = R_A$ and a decomposition of the identity into N mutually orthogonal nonzero projections E_j, $I = \sum_{j=1}^N E_j$, for which $\sum_{j=1}^N E_jVAV^*E_j = I$, the convergence of the series being in the strong operator topology if $N = \infty$.*

(ii) *A is the sum of N nonzero projections, the convergence of the series being in the strong operator topology if $N = \infty$, and if M is semifinite, then $\tau(A) = \tau(I)$.*

Proof.

(i) $\implies$ (ii) For every j, let $W_j := E_jVA^{\frac{1}{2}}$ and let $P_j := W_j^*W_j$. Since

$$W_jW_j^* = E_jVAV^*E_j = E_j\sum_{i=1}^N E_iVAV^*E_iE_j = E_j,$$

we see that W_j is a partial isometry, and hence, P_j is a projection and $P_j \sim E_j$ for every j. Then

$$\sum_{j=1}^{N} P_j = \sum_{j=1}^{N} A^{\frac{1}{2}}V^*E_jVA^{\frac{1}{2}} = A^{\frac{1}{2}}V^*\big(\sum_{j=1}^{N} E_j\big)VA^{\frac{1}{2}} = A^{\frac{1}{2}}V^*VA^{\frac{1}{2}} = A^{\frac{1}{2}}R_AA^{\frac{1}{2}} = A$$

and if $N = \infty$ the series $\sum_{j=1}^{N} E_j$ and hence the series $\sum_{j=1}^{N} P_j$ converge in the strong operator topology. Furthermore, if M is semifinite, by the normality of the trace τ we have

$$\tau(A) = \sum_{j=1}^{N} \tau(P_j) = \sum_{j=1}^{N} \tau(E_j) = \tau(I).$$

(ii) $\implies$ (i) Let $A = \sum_{j=1}^{N} P_j$ where $\{P_j\}$ are nonzero projections. First, we decompose the identity $I = \sum_{j=1}^{N} E_j$ into N mutually orthogonal projections $E_j \sim P_j$. This is immediate if all the projections P_j are infinite, and hence so is I, because then we can decompose I into N mutually orthogonal infinite projections and all infinite projections are equivalent by the assumption that M is σ-finite. Assume henceforth that M is semifinite and that $\Lambda := \{j \mid \tau(P_j) < \infty\} \neq \emptyset$ and let Λ' be its (possibly empty) complement. Then

$$\tau(I) = \tau(A) = \sum_{j=1}^{N} \tau(P_j) \geq \sum_{j\in\Lambda} \tau(P_j).$$

Whether M is of type II or it is of type I and then $\sum_{j\in\Lambda} \tau(P_j) \in \mathbb{N} \cup \{\infty\}$, there exists a projection F with $\tau(F) = \sum_{j\in\Lambda} \tau(P_j)$. Then it is routine to find mutually orthogonal projections $E_j \leq F$ with $\tau(E_j) = \tau(P_j)$ for every $j \in \Lambda$. Let $E := \sum_{j\in\Lambda} E_j$. Then $E \leq F$ and $\tau(E) = \tau(F) = \tau(I)$. We now consider three cases.

In the first case, assume that $\tau(I) < \infty$. Then $\tau(I - E) = 0$, hence $E = I$, and we are done.

In the second case, assume that $\tau(I) = \infty$ and $\Lambda' = \emptyset$. Then E is an infinite projection, hence there is an isometry W for which $WW^* = E$. Set $E_j' = W^*E_jW$. Then $E_j' \sim E_j \sim P_j$ for every j and $I = \sum_{j=1}^{N} E_j'$ provides the required decomposition.

In the third case, assume that $\tau(I) = \infty$ and $\Lambda' \neq \emptyset$. Modify if necessary F so that $I - F$ is infinite and hence so is $I - E \geq I - F$. Then decompose $I - E$ into $\operatorname{card} \Lambda'$ mutually orthogonal infinite projections E_j, $I - E = \sum_{j\in\Lambda'} E_j$. Since $P_j \sim E_j$ for all $j \in \Lambda'$, $I = \sum_{j\in\Lambda} E_j + \sum_{j\in\Lambda'} E_j = \sum_{j=1}^{N} E_j$ provides in this case too the required decomposition.

Now choose partial isometries W_j with $P_j = W_j^*W_j$ and $E_j = W_jW_j^*$. If $N < \infty$, set $B := \sum_{j=1}^{N} W_j$. If $N = \infty$ and $m > n$, then

$$\big(\sum_{j=m}^{n} W_j\big)^*\big(\sum_{j=m}^{n} W_j\big) = \sum_{i,j=m}^{n} V_i^*W_j = \sum_{j=m}^{n} W_j^*W_j = \sum_{j=m}^{n} P_j. \tag{8}$$

Thus, by the strong (and hence the weak) convergence of the series $\sum_{j=1}^{\infty} P_j$, we see that the series $\sum_{j=1}^{\infty} W_j$ is strongly Cauchy and hence converges in the strong operator topology. Again, call its sum B. By the same computation as in (8), we have $B^*B = \sum_{j=1}^{N} P_j = A$. Let $B = VA^{\frac{1}{2}}$ be the polar decomposition of B. Then $V^*V = R_A$ and $BB^* = VAV^*$. Moreover,

$E_jB = W_j$ for every j, thus

$$\sum_{j=1}^{N} E_jVAV^*E_j = \sum_{j=1}^{N} E_jBB^*E_j = \sum_{j=1}^{N} W_jW_j^* = \sum_{j=1}^{N} E_j = I.$$

□

Lemma 3.2. *Let $A \in M^+$ be a strong sum of projections.*
(i) *Either $||A|| > 1$ (equivalently, $A_+ \neq 0$) or A is a projection.*
(ii) *If M is semifinite, then $\tau(R_A) \leq \tau(A)$.*

Proof.
(i) Obvious, since if P, Q are projections, then $||P+Q|| = 1$ if and only if $PQ = 0$ if and only if $P+Q$ is a projection.
(ii) Let $A = \sum_{j=1}^{N} P_j$ with P_j nonzero projections and $N \in \mathbb{N} \cup \{\infty\}$ and assume without loss of generality that $\tau(A) < \infty$. By Kaplansky's parallelogram law, [8, Theorem, 6.1.7], for every integer $n \leq N$ we have

$$\tau\big(\bigvee_{j=1}^{n} P_j\big) \leq \sum_{j=1}^{n} \tau(P_j) \leq \tau(A).$$

If $N < \infty$, then $R_A = \bigvee_{j=1}^{N} P_j$ and we are done. If $N = \infty$, then $\bigvee_{j=1}^{n} P_j \uparrow R_A$ and by the normality of τ, $\tau\big(\bigvee_{j=1}^{n} P_j\big) \uparrow \tau(R_A)$. Thus also $\tau(R_A) \leq \tau(A)$.

□

Theorem 3.3. *Assume that $A \in M$ is a strong sum of projections. Then*
(i) *If M is of type I, then Tr $(A_+) \geq$ Tr (A_-) and either Tr $(A_+) = \infty$ or Tr $(A_+) < \infty$ and Tr $(A_+) -$ Tr $(A_-) \in \mathbb{N} \cup \{0\}$.*
(ii) *If M is of type II, then $\tau(A_+) \geq \tau(A_-)$.*
(iii) *If M is of type III, then either $||A|| > 1$ (equivalently, $A_+ \neq 0$) or A is a projection.*

Proof.
(iii) is given by Lemma 3.2 (i), so assume henceforth that M is semifinite. Let $A = \sum_{j=1}^{N} P_j$ with P_j nonzero projections and $N \in \mathbb{N} \cup \{\infty\}$.

Assume first that $\tau(R_A) < \infty$ and hence also $\tau(A) < \infty$ and $\tau(A_-) < \infty$. Then by (1) and by Lemma 3.2 we have $\tau(A_+) - \tau(A_-) = \tau(A) - \tau(R_A) \geq 0$. Moreover, if M is of type I, then $N < \infty$ and both $\tau(A)$ and $\tau(R_A)$ are positive integers, which proves the integrality condition in (i) for the case when $\tau(R_A) < \infty$ (see also Corollary 2.5).

Now assume that $\tau(R_A) = \infty$ and assume furthermore that $\tau(A_+) < \infty$. Obviously, $\tau(I) = \infty$ and by Lemma 3.2 , $\tau(A) = \infty$, hence $\tau(A) = \tau(I)$. Thus by Proposition 3.1 there is a partial isometry V with $V^*V = R_A$ and a decomposition of the identity $I = \sum_{j=1}^{N} E_j$ into N mutually orthogonal projections E_j for which $\sum_{j=1}^{N} E_jVAV^*E_j = I$. Recall that the map

$$M \ni X \to \Phi(X) := \sum_{j=1}^{N} E_jXE_j \in M$$

is linear, positive, unital, faithful, and in case M is semifinite, it is also trace preserving. Then we have by (1) that

$$I = \Phi(VAV^*) = \Phi(VA_+V^*) - \Phi(VA_-V^*) + \Phi(VR_AV^*).$$

It follows from $V^*V = R_A$ that

$$\Phi(VA_+V^*) = \Phi(VA_-V^*) + \Phi(I - VV^*). \tag{9}$$

and hence

$$\tau(VA_-V^*) = \tau(\Phi(VA_-V^*)) \leq \tau(\Phi(VA_+V^*)) = \tau(VA_+V^*) = \tau(R_AA_+) = \tau(A_+) < \infty.$$

But then,

$$\tau(A_-) = \tau(R_AA_-R_A) = \tau(V^*VA_-V^*V) = \tau(VV^*VA_-V^*) = \tau(VA_-V^*).$$

This concludes the proof of the case when M is of type II. If M is of type I and $\mathrm{Tr}\ (A_+) < \infty$, it follows from (9) and the above computations that

$$\mathrm{Tr}\ (A_+) = \mathrm{Tr}\ (A_-) + \mathrm{Tr}\ (\Phi(I - VV^*)) = \mathrm{Tr}\ (A_-) + \mathrm{Tr}\ (I - VV^*).$$

This shows that $\mathrm{Tr}\ (I - VV^*) < \infty$, i.e., $I - VV^*$ is a finite projection, and therefore $\mathrm{Tr}\ (I - VV^*) \in \mathbb{N} \cup \{0\}$.

□

Remark 3.4. *Notice that if M is semifinite and $A \in M^+$, then*

$$\tau(A_+) \geq \tau(A_-) \Longrightarrow \tau(A) \geq \tau(R_A) \not\Longrightarrow \tau(A_+) \geq \tau(A_-).$$

However, if $\tau(R_A) < \infty$, then $\tau(A_+) \geq \tau(A_-) \Longleftrightarrow \tau(A) \geq \tau(R_A)$.

Remark 3.5. *In the case of $M = B(H)$, let $\mathcal{V}(A) := \{VAV^* \mid V^*V = R_A\}$ be the partial isometry orbit of A and let E denote the (unique) normal conditional expectation on the diagonal masa of $B(H)$ (according to a fixed orthonormal basis). Then Proposition 3.1 states that a positive operator $A \in B(H)$ with infinite trace is a strong sum of rank-one projections if and only if $I \in E(\mathcal{V}(A))$. When A is also invertible, this is a special case of* [1, Proposition 4.5]. *In the case of compact operators, the diagonals of the partial isometry orbit are characterized in terms of majorization of sequences by the infinite dimensional Schur-Horn theorem obtained in* [13]. *The set $E(\mathcal{V}(A))$ is further studied in in* [15] *for the case of positive not necessarily compact operators.*

In the case of $M = B(H)$, an application of Proposition 3.1 together with a modification of the proof of Theorem 3.3 (i) provides an alternative proof of the necessity of Kadison's integrality condition in [9, Theorem 15] that characterizes the diagonals of infinite co-infinite projections and identifies explicitly the integer as the difference of the traces of two projections.

Corollary 3.6. [9, Theorem 15] *Let $P \in B(H)$ be an infinite, co-infinite projection, let e_n be an orthonormal basis, let $c_n := (Pe_n, e_n)$, and assume that $\sum\{c_n \mid c_n \leq \frac{1}{2}\} < \infty$ and $\sum\{1 - c_n \mid c_n > \frac{1}{2}\} < \infty$. Then*

$$\sum\{1 - c_n \mid c_n > \frac{1}{2}\} - \sum\{c_n \mid c_n \leq \frac{1}{2}\} \in \mathbb{Z}.$$

Proof. Let W be an isometry with $P = WW^*$. Define

$$w_n = \begin{cases} \frac{1}{\sqrt{c_n}} W^* e_n & \text{if } c_n \neq 0 \\ e_1 & \text{if } c_n = 0 \end{cases} \qquad \text{and} \qquad P_n := w_n \otimes w_n$$

Then $||w_n|| = 1$ for every n and hence P_n are rank-one projections. A simple computation shows that $I = \sum_n c_n P_n$, with the series converging in the strong operator topology. Define

$$T_+ := \sum\{(1-c_n)P_n \mid c_n > \frac{1}{2}\}, \qquad T_- := \sum\{c_nP_n \mid c_n \leq \frac{1}{2}\}, \qquad \textit{and} \qquad T := T_+ - T_-.$$

Then T_+, T_-, and hence T are trace class operators and

$$\text{Tr } T = \sum\{1-c_n \mid c_n > \frac{1}{2}\} - \sum\{c_n \mid c_n \leq \frac{1}{2}\}.$$

Since $\sum\{c_nP_n \mid c_n > \frac{1}{2}\}$ and $\sum\{(1-c_n)P_n \mid c_n > \frac{1}{2}\}$ both converge in the strong operator topology, it follows that also $\sum\{P_n \mid c_n > \frac{1}{2}\}$ converges in the strong operator topology. Set $A := \sum\{P_n \mid c_n > \frac{1}{2}\}$. By Proposition 3.1 there is a partial isometry V with $V^*V = R_A$ for which $E(VAV^*) = I$, where E is the conditional expectation on the atomic masa (the operation of taking the main diagonal).

Since $I = A - T$, we have $VV^* = VAV^* - VTV^*$ and thus

$$E(VV^*) = E(VAV^*) - E(VTV^*) = I - E(VTV^*) = E(I) - E(VTV^*).$$

Thus $E(VTV^*) = E(I - VV^*)$ and hence

$$\text{Tr } (V^*VT) = \text{Tr } (VTV^*) = \text{Tr } (I - VV^*) \in \mathbb{Z}$$

since T is trace-class and VV^* and hence $I - VV^*$ are projections. On the other hand,

$$(V^*V)^{\perp}T = (V^*V)^{\perp}(A - I) = -(V^*V)^{\perp},$$

hence $\text{Tr } ((V^*V)^{\perp}T) \in \mathbb{Z}$, and thus

$$\text{Tr } (T) = \text{Tr } (V^*VT) + \text{Tr } ((V^*V)^{\perp}T) \in \mathbb{Z}.$$

□

4. B(H): THE FINITE TRACE CASE

In this section we will prove that if $A \in B(H)^+$ and $\text{Tr } (A_-) \leq \text{Tr } (A_+) < \infty$ and $\text{Tr } (A_+) - \text{Tr } (A_-) \in \mathbb{N} \cup \{0\}$, then A is a strong sum of projections. Since the trace-class operators A_+ and A_- are diagonalizable and have orthogonal supports, then by (1), A too is diagonalizable. As in Lemma 2.3 , let us denote the eigenvalues of A which are larger than 1, if any, by $1 + \mu_j$ and those are less or equal than 1, if any, by $1 - \lambda_i$, i.e., set

$$A := \sum_{j=1}^{N}(1+\mu_j)(e_j \otimes e_j) + \sum_{i=1}^{K}(1-\lambda_i)(f_i \otimes f_i)$$

where $N, K \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, the unit vectors e_j, f_i are mutually orthogonal, $\mu_j > 0$, and $0 \leq \lambda_i \leq 1$ for all i and j. Notice that the series, if infinite, converge in the strong operator topology. Of course, it would be equivalent to assume that $\mu_j \geq 0$ and $0 < \lambda_i < 1$ for all i and j. Thus $A_+ = \sum_{j=1}^{N} \mu_j (e_j \otimes e_j)$ and $A_- = \sum_{i=1}^{K} \lambda_i (f_i \otimes f_i)$ and hence

$$\sum_{j=1}^{N} \mu_j - \sum_{i=1}^{K} \lambda_i \in \mathbb{N} \cup \{0\}.$$

Here too we adopt the convention to set a series $\sum_{i=1}^{0}$ as zero, e.g., by $K = 0$ we mean that A has no non-negative eigenvalues less or equal than 1, and hence, $A_- = 0$; similarly for $N = 0$.

Our proof will depend on iterative applications of Lemma 2.1. Since we will focus on infinite rank operators, i.e., on the case when $N+K=\infty$, the process will not terminate as in Lemma 2.3 after a finite number of steps and the crux of the proofs will be to establish strong convergence. This will be illustrated by the following lemma which handles two key special cases.

Lemma 4.1. *Let $\{g_o, g_1, \dots\}$ be mutually orthogonal unit vectors.*
(i) *Let $A = (1-\lambda)(g_o \otimes g_o) + \sum_{j=1}^{\infty}(1+\mu_j)(g_j \otimes g_j)$ where $\mu_j > 0$, $0 \le \lambda \le 1$, for all j and $\lambda = \sum_{j=1}^{\infty} \mu_j$. Then A is a strong sum of projections.*
(ii) *Let $A = (1+\mu)(g_o \otimes g_o) + \sum_{j=1}^{\infty}(1-\lambda_j)(g_j \otimes g_j)$ where $\mu > 0$, $0 \le \lambda_j \le 1$ for all j and $\mu = \sum_{j=1}^{\infty} \lambda_j$. Then A is a strong sum of projections.*

Proof.
(i) If $\lambda = 0$, then $\mu_j = 0$ for all j and hence A is already a projection. Thus assume that $\lambda \neq 0$. Define

$$\delta_j := \begin{cases} -\lambda & j = 1 \\ \sum_{i=1}^{j-1} \mu_i - \lambda & j > 1. \end{cases}$$

Then δ_j increases strictly to 0, so we can also define

$$\sigma_j := \begin{cases} 0 & j = 1 \\ \frac{(1+\delta_{j-1})\delta_{j-1}}{(1+\delta_j)(2\delta_{j-1}-\delta_j)} & j > 1. \end{cases} \tag{10}$$

Then for every $j > 1$, $\sigma_j > 0$ and also

$$1 - \sigma_j = \frac{(1+\delta_j - \delta_{j-1})(\delta_{j-1} - \delta_j)}{(1+\delta_j)(2\delta_{j-1} - \delta_j)} > 0. \tag{11}$$

Define also

$$v_j := \begin{cases} g_o & j = 1 \\ \sqrt{\sigma_j}v_{j-1} + \sqrt{1-\sigma_j}g_{j-1} & j > 1. \end{cases} \tag{12}$$

Solving this recurrence relation, we get

$$v_j = \sum_{k=0}^{j-2} \Big(\sqrt{1-\sigma_{k+1}} \prod_{i=k+2}^{j} \sqrt{\sigma_i} \Big) g_k + \sqrt{1-\sigma_j} g_{j-1}. \tag{13}$$

We claim that there is a sequence of rank-one projections P_j for which

$$(1-\lambda)(g_o \otimes g_o) + \sum_{j=1}^{n}(1+\mu_j)(g_j \otimes g_j) = \sum_{j=1}^{n} P_j + (1+\delta_{n+1})(v_{n+1} \otimes v_{n+1}) \tag{14}$$

for every n. By Lemma 2.1

$$(1-\lambda)(g_o \otimes g_o) + (1+\mu_1)(g_1 \otimes g_1) = P_1 + (1+\mu_1-\lambda)(v \otimes v) = P_1 + (1+\delta_2)(v \otimes v)$$

where P_1 is a rank-one projection and by (3) and (2), $v = \sqrt{\nu}g_o + \sqrt{1-\nu}g_1$ and

$$\nu = \frac{(1-\lambda)\lambda}{(1+\mu_1-\lambda)(\mu_1+\lambda)} = \frac{(1+\delta_1)(-\delta_1)}{(1+\delta_2)(\delta_2 - 2\delta_1)} = \sigma_2.$$

Thus $v = v_2$ and hence (14) is satisfied for $n = 1$. Assume that (14) is satisfied for $n-1$. Then

$$\begin{aligned}
&(1-\lambda)(g_o \otimes g_o) + \sum_{j=1}^{n}(1+\mu_j)(g_j \otimes g_j) = \\
&\qquad = \sum_{j=1}^{n-1} P_j + (1+\delta_n)(v_n \otimes v_n) + (1+\mu_n)(g_n \otimes g_n) && \text{(by the induction hypothesis)} \\
&\qquad = \sum_{j=1}^{n-1} P_j + P_n + (1+\mu_n+\delta_n)(v \otimes v) && \text{(by Lemma 2.1)} \\
&\qquad = \sum_{j=1}^{n} P_j + (1+\delta_{n+1})(v \otimes v) && \text{(by the definition of } \delta)
\end{aligned}$$

where P_n is a rank-one projection, and by (3) and (2), $v = \sqrt{\nu} v_n + \sqrt{1-\nu} g_n$ and

$$\nu = \frac{(1+\delta_n))(-\delta_n)}{(1+\mu_n+\delta_n)(\mu_n-\delta_n)} = \frac{(1+\delta_n)\delta_n}{(1+\delta_{n+1})(2\delta_n-\delta_{n+1})} = \sigma_{n+1}.$$

Hence $v = v_{n+1}$ and thus (14) is satisfied for n. Thus for every n,

$$A = \sum_{i=1}^{n} P_i + (1+\delta_{n+1})(v_{n+1} \otimes v_{n+1}) + \sum_{j=n+1}^{\infty}(1+\mu_j)(g_j \otimes g_j).$$

Since $\sum_{j=n+1}^{\infty}(1+\mu_j)(g_j \otimes g_j) \underset{s}{\rightarrow} 0$, to prove that $A = \sum_{i=1}^{\infty} P_i$ where the convergence is in the strong topology (and hence, to establish the thesis), we need to show that $v_{n+1} \otimes v_{n+1} \underset{s}{\rightarrow} 0$, or, equivalently, that $v_j \to 0$ weakly. Since $v_j \in \operatorname{span}\{g_i\}$, it is enough to show that $(v_j, g_q) \to 0$ for every $q \in \mathbb{N} \cup \{0\}$. Indeed, for every $j > q+1$, we have from (13) that

$$(v_j, g_q) = \sqrt{1-\sigma_{q+1}} \prod_{i=q+2}^{j} \sqrt{\sigma_i}.$$

Thus it is enough to show that $\prod_{i=2}^{j} \sigma_i \to 0$, or, equivalently, that $\sum_{i=2}^{\infty}(1-\sigma_i) = \infty$. By (11) and since $\delta_{j-1} < \delta_j < 0$ we have

$$1-\sigma_j = \frac{(1-\delta_{j-1}+\delta_j)(\delta_{j-1}-\delta_j)}{(1+\delta_j)(2\delta_{j-1}-\delta_j)} > \frac{\delta_{j-1}-\delta_j}{2\delta_{j-1}-\delta_j} > \frac{1}{2}\frac{\delta_{j-1}-\delta_j}{\delta_{j-1}} > 0. \tag{15}$$

Since $\delta_j \uparrow 0$, for every $n > m$,

$$\sum_{i=m+1}^{n} \frac{\delta_{i-1}-\delta_i}{\delta_{i-1}} \geq \sum_{i=m+1}^{n} \frac{\delta_{i-1}-\delta_i}{\delta_m} = \frac{\delta_m-\delta_n}{\delta_m} = 1 - \frac{\delta_n}{\delta_m},$$

whence $\sum_{i=m+1}^{\infty} \frac{\delta_{i-1}-\delta_i}{\delta_{i-1}} > \frac{1}{2}$ for every m. As a consequence, $\sum_{j=2}^{\infty} \frac{\delta_{i-1}-\delta_i}{\delta_{i-1}} = \infty$, and thus, $\sum_{j=2}^{\infty}(1-\sigma_j) = \infty$, which completes the proof for this case.

(ii) Let $k := \operatorname{card}\{j \mid \lambda_j = 1\}$. By passing to

$$\begin{aligned}
A' :&= A - k(g_o \otimes g_o) - \sum\{g_j \otimes g_j \mid \lambda_j = 0\} \\
&= (1+\mu-k)(g_o \otimes g_o) + \sum\{(1-\lambda_j)(g_j \otimes g_j) \mid 0 < \lambda_j < 1\},
\end{aligned}$$

we can assume without loss of generality that $0 < \lambda_j < 1$ for all j. Define

$$\delta_j := \begin{cases} \mu & j = 1 \\ \mu - \sum_{i=1}^{j-1} \lambda_i & j > 1 \end{cases}.$$

Then $\delta_j \downarrow 0$. Let σ_j and v_j be defined by (10) and (12) respectively. We claim that there is a sequence of rank-one projections P_j for which

$$(1+\mu)(g_o \otimes g_o) + \sum_{j=1}^{n}(1-\lambda_j)(g_j \otimes g_j) = \sum_{j=1}^{n} P_j + (1+\delta_{n+1})(v_{n+1} \otimes v_{n+1}) \tag{16}$$

for every n.

Apply Lemma 4.2 to obtain

$$(1-\lambda_1)(g_1 \otimes g_1) + (1+\mu)(g_o \otimes g_o) = P_1 + (1+\delta_2)(v \otimes v)$$

where P_1 is a rank-one projection and by (3), (2), and (11), $v = \sqrt{\nu} g_1 + \sqrt{1-\nu} g_o$ and

$$\nu = \frac{(1-\lambda_1)\lambda_1}{(1+\mu-\lambda_1)(\mu+\lambda_1)} = \frac{(1+\delta_2-\delta_1)(\delta_1-\delta_2)}{(1+\delta_2)(2\delta_1-\delta_2)} = 1-\sigma_2.$$

Thus $v = v_2$ and (16) holds for $n = 1$. The inductive proof of the claim then proceeds as in part (i). Thus

$$A = \sum_{j=1}^{n} P_j + (1+\delta_{n+1})(v_{n+1} \otimes v_{n+1}) + \sum_{j=n+1}^{\infty} (1-\lambda_j)(g_j \otimes g_j),$$

and hence, to prove that $A = \sum_{j=1}^{\infty} P_j$ we need to show that $v_j \to 0$ weakly. Again, by (13) it suffices to show that $\sum_{j=2}^{\infty}(1-\sigma_j) = \infty$. The only difference from the proof of part (i) is that the inequality used in (15) does no longer hold since $\delta_j > 0$. However, since $\delta_j \to 0$, we have, for j large enough,

$$1-\sigma_j = \frac{(1-\delta_{j-1}+\delta_j)(\delta_{j-1}-\delta_j)}{(1+\delta_j)(2\delta_{j-1}-\delta_j)} > \frac{1}{2}\frac{\delta_{j-1}-\delta_j}{2\delta_{j-1}-\delta_j} > \frac{1}{4}\frac{\delta_{j-1}-\delta_j}{\delta_{j-1}} > 0.$$

Then the same argument as in part (i) proves the claim.

□

The next special case is also based on iterated applications of Lemma 2.1 and shares part of the construction with the previous lemma, but with a different proof of the weak convergence of the vector sequence.

Lemma 4.2. *Let $A = \sum_{i=1}^{\infty}(1+\mu_i)(e_i \otimes e_i) + \sum_{i=1}^{\infty}(1-\lambda_i)(f_i \otimes f_i)$ where $\{e_i, f_i\}$ are mutually orthogonal unit vectors, $\mu_i > 0$, $0 < \lambda_i < 1$, for all i, $\sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\infty} \mu_i < \infty$, and $\sum_{i=1}^{m} \lambda_i \neq \sum_{i=1}^{n} \mu_i$ for every $n, m \in \mathbb{N}$. Then A is a strong sum of projections.*

Proof. Since by hypothesis $\lambda_1 \neq \mu_1$, we assume that $\lambda_1 > \mu_1$ and leave to the reader the similar proof for the case when $\lambda_1 < \mu_1$. Since $\lambda_1 < \sum_{j=1}^{\infty} \mu_j$, there is a smallest integer n_1 for which $\lambda_1 < \sum_{j=1}^{n_1} \mu_j$. Similarly, there is a smallest integer m_1 for which $\sum_{j=1}^{m_1} \lambda_j > \sum_{j=1}^{n_1} \mu_j$.

From here we obtain recursively the strictly increasing integer sequences $\{m_k\}$, $\{n_k\}$, starting with $n_o = 0$, $m_o = 1$, for which

$$\sum_{j=1}^{n_{k-1}} \mu_j \le \sum_{j=1}^{n_k - 1} \mu_j < \sum_{j=1}^{m_{k-1}} \lambda_j \le \sum_{j=1}^{m_k - 1} \lambda_j < \sum_{j=1}^{n_k} \mu_j \le \sum_{j=1}^{n_{k+1}-1} \mu_j < \sum_{j=1}^{m_k} \lambda_j. \tag{17}$$

Set

$$A_j := \begin{cases} (1-\lambda_1)(f_1 \otimes f_1) & j = 1 \\ (1-\lambda_{j-n_k})(f_{j-n_k} \otimes f_{j-n_k}) & m_{k-1} + n_k < j \le m_k + n_k \\ (1+\mu_{j-m_k})(e_{j-m_k} \otimes e_{j-m_k}) & m_k + n_k < j \le m_k + n_{k+1} \end{cases}$$

Since A is the sum of two series which converge unconditionally, we can rearrange its summands to obtain $A = \sum_{i=1}^{\infty} A_i$ (in the strong topology.) Explicitly, for $j > 1$,

$$\sum_{i=1}^{j} A_i = \begin{cases} \sum_{i=1}^{n_k}(1+\mu_i)(e_i \otimes e_i) + \sum_{i=1}^{j-n_k}(1-\lambda_i)(f_i \otimes f_i) & m_{k-1} + n_k \le j \le m_k + n_k \\ \sum_{i=1}^{j-m_k}(1+\mu_i)(e_i \otimes e_i) + \sum_{i=1}^{m_k}(1-\lambda_i)(f_i \otimes f_i) & m_k + n_k \le j \le m_k + n_{k+1}. \end{cases}$$

Define

$$\delta_j = \begin{cases} -\lambda_1 & j = 1 \\ \sum_{i=1}^{n_k} \mu_i - \sum_{i=1}^{j-n_k} \lambda_i & m_{k-1} + n_k < j \le m_k + n_k \\ \sum_{i=1}^{j-m_k} \mu_i - \sum_{i=1}^{m_k} \lambda_i & m_k + n_k < j \le m_k + n_{k+1}. \end{cases}$$

Then from (17) we have

$$\begin{aligned} &\delta_j > 0 \qquad m_{k-1} + n_k \le j < m_k + n_k \\ &\delta_j < 0 \qquad m_k + n_k \le j < m_k + n_{k+1} \end{aligned} \tag{18}$$

and

$$\delta_j - \delta_{j-1} = \begin{cases} -\lambda_{j-n_k} < 0 & m_{k-1} + n_k < j \le m_k + n_k \\ \ \ \mu_{j-m_k} > 0 & m_k + n_k < j \le m_k + n_{k+1}. \end{cases} \tag{19}$$

Thus

$$\begin{aligned} \min\{\delta_j \mid m_{k-1} + n_k \le j \le m_k + n_{k+1}\} &= \delta_{n_k + m_k} && \text{(by (19))} \\ &= \delta_{n_k + m_k - 1} - \lambda_{m_k} && \text{(by (19))} \\ &> -\lambda_{m_k} && \text{(by (18))} \\ &> -1 && \text{(by hypothesis).} \end{aligned} \tag{20}$$

Moreover,

$$\begin{aligned} &2\delta_{j-1} - \delta_j = \delta_{j-1} + \lambda_{j-n_k} > \delta_{j-1} > 0 \qquad m_{k-1} + n_k < j \le m_k + n_k \\ &2\delta_{j-1} - \delta_j = \delta_{j-1} - \mu_{j-m_k} < \delta_{j-1} < 0 \quad\ m_k + n_k < j \le m_k + n_{k+1}. \end{aligned} \tag{21}$$

Define the sequence σ_j as in (10). From (18), (19), and (21), we see that for every j, δ_{j-1}, $\delta_{j-1} - \delta_j$, and $2\delta_{j-1} - \delta_j$ have the same sign. Since furthermore $1 + \delta_j > 0$ by (20) and $1 + \delta_j - \delta_{j-1} > 0$ by (19) and (20), it follows that $0 < \sigma_j < 1$.

Now let $J_k := m_k + n_{k+1}$. Then we have by (18) that $\delta_{J_k-1} < 0 < \delta_{J_k}$, by (20) that $1+\delta_{J_k} > 0$, and hence $\frac{1+\delta_{J_k-1}}{1+\delta_{J_k}} < 1$. As $2\delta_{J_k-1}-\delta_{J_k} < 0$ by (21), we also have $0 < \frac{\delta_{J_k-1}}{2\delta_{J_k-1}-\delta_j} < \frac{1}{2}$ and thus

$$\sigma_{m_k+n_{k+1}} < \frac{1}{2} = \sigma_{J_k}. \tag{22}$$

Having concluded these preliminary computations, we define recursively the sequence of unit vectors

$$v_j = \begin{cases} f_1 & j = 1 \\ \sqrt{\sigma_j}v_{j-1} + \sqrt{1-\sigma_j}f_{j-n_k} & m_{k-1}+n_k < j \le m_k + n_k \\ \sqrt{\sigma_j}v_{j-1} + \sqrt{1-\sigma_j}e_{j-m_k} & m_k + n_k < j \le m_k + n_{k+1}. \end{cases} \tag{23}$$

Notice that

$$v_j \in \begin{cases} \operatorname{span}\{f_1,\dots,f_{j-n_k},e_1,\dots,e_{n_k}\} & m_{k-1}+n_k \le j \le m_k+n_k \\ \operatorname{span}\{f_1,\dots,f_{m_k},e_1,\dots,e_{j-m_k}\} & m_k+n_k \le j \le m_k+n_{k+1}. \end{cases} \tag{24}$$

Now we claim that there is a sequence of rank-one projections P_j for which

$$\sum_{j=1}^{n} A_j = \sum_{j=1}^{n-1} P_j + (1+\delta_n)(v_n \otimes v_n) \quad \text{for } n \ge 2. \tag{25}$$

By Lemma 2.1

$$\begin{aligned} A_1 + A_2 &= (1-\lambda_1)(f_1\otimes f_1) + (1+\mu_1)(e_1\otimes e_1) && \text{(by definition)} \\ &= P_1 + (1+\mu_1-\lambda_1)(v\otimes v) && \text{(by Lemma 2.1)} \\ &= P_1 + (1+\delta_2)(v\otimes v) && \text{(by definition)} \end{aligned}$$

where P_1 is a rank-one projection and by (3), (2), $v = \sqrt{\nu}f_1 + \sqrt{1-\nu}e_1$ and

$$\nu = \frac{(1-\lambda_1)\lambda_1}{(1+\mu_1-\lambda_1)(\mu_1+\lambda_1)} = \sigma_2$$

and hence $v = v_2$.

Since $\delta_2 < 0$ by (18) and $v_2 \perp e_2$, we can apply Lemma 2.1 to

$$(1+\delta_2)(v_2\otimes v_2) + (1+\mu_2)(e_2\otimes e_2)$$

and continue the process. Assume the construction up to $j-1$, where $m_{k-1}+n_k < j \le m_k+n_k$ for some k. Then

$$\begin{aligned} \sum_{i=1}^{j} A_i &= \sum_{i=1}^{j-1} A_i + (1-\lambda_{j-n_k})(f_{j-n_k}\otimes f_{j-n_k}) \\ &= \sum_{1-1}^{j-2} P_i + (1+\delta_{j-1})(v_{j-1}\otimes v_{j-1}) + (1-\lambda_{j-n_k})(f_{j-n_k}\otimes f_{j-n_k}). \end{aligned}$$

Now $v_{j-1} \perp f_{j-n_k}$ by (24) and $\delta_{j-1} > 0$ by (18), so we can apply Lemma 2.1 and obtain

$$(1+\delta_{j-1})(v_{j-1}\otimes v_{j-1}) + (1-\lambda_{j-n_k})(f_{j-n_k}\otimes f_{j-n_k}) = P_{j-1} + (1+\delta_{j-1}-\lambda_{j-n_k})(v\otimes v)$$

where P_{j-1} is a rank-one projection, and by (3), (2), $v = \sqrt{\nu} f_{j-n_k} + \sqrt{1-\nu}\, v_{j-1}$ and

$$\begin{aligned}
\nu &= \frac{(1-\lambda_{j-n_k})\lambda_{j-n_k}}{(1+\delta_{j-1}-\lambda_{j-n_k})(\delta_{j-1}+\lambda_{j-n_k})} && \text{(by Lemma 2.1)}\\
&= \frac{(1+\delta_j-\delta_{j-1})(\delta_{j-1}-\delta_j)}{(1+\delta_j)(2\delta_{j-1}-\delta_j)} && \text{(since } \delta_j = \delta_{j-1}-\lambda_{j-n_k} \text{ by (19))}\\
&= 1-\sigma_j && \text{(by (11))}
\end{aligned}$$

But then, $v = v_j$ and since also $\delta_j = \delta_{j-1} - \lambda_{j-n_k}$, we see that (25) is satisfied for j. We leave to the reader the similar proof for the case when $m_k + n_k < j \leq m_k + n_{k+1}$ for some k. We thus have for all n

$$A - \sum_{i=1}^{n} P_i = (1+\delta_{n+1})(v_{n+1} \otimes v_{n+1}) + \sum_{i=n+1}^{\infty} A_i,$$

Since $\sum_{i=n+1}^{\infty} A_i \underset{s}{\rightarrow} 0$, as in the proof of Lemma 4.1, in order to prove that $A = \sum_{i=1}^{\infty} P_i$ in the strong topology, it suffices to show that the projections $v_j \otimes v_j \underset{s}{\rightarrow} 0$, or, equivalently, that the sequence of unit vectors $v_j \underset{w}{\rightarrow} 0$. Since all $v_j \in \operatorname{span}\{f_i, e_i\}$, it suffices to prove that $(v_j, f_q) \rightarrow 0$ and $(v_j, e_q) \rightarrow 0$ for all q.

Fix $q \in \mathbb{N}$ and choose h such that $m_h \geq q$ and $n_h \geq q$ and let $w = v_{m_h+n_h}$. From (23) we have

$$v_{m_h+n_h+1} - \sqrt{\sigma_{m_h+n_h+1}}w = \sqrt{1-\sigma_{m_h+n_h+1}}e_{n_h+1} \in \{f_1, \ldots, f_{m_h}, e_1, \ldots, e_{n_h}\}^{\perp}.$$

Iterating,

$$v_j - \Big(\prod_{i=m_h+n_h+1}^{j} \sqrt{\sigma_i}\Big) w \in \{f_1, \ldots, f_{m_h}, e_1, \ldots, e_{n_h}\}^{\perp} \quad \text{for every } j > m_h + n_h.$$

In particular for every $j > m_h + n_h$,

$$(v_j, f_q) = \Big(\prod_{i=m_h+n_h+1}^{j} \sqrt{\sigma_i}\Big)\,(w, f_q) \quad \text{and} \quad (v_j, e_q) = \Big(\prod_{i=m_h+n_h+1}^{j} \sqrt{\sigma_i}\Big)\,(w, e_q).$$

Since $0 < \sigma_i < 1$ for all i by (10) and (11) and $\sigma_i < \frac{1}{2}$ infinitely often by (22), we see that $\prod_{i=m_h+n_h+1}^{j} \sqrt{\sigma_i} \rightarrow 0$ and hence $v_j \rightarrow 0$ weakly, which concludes the proof. □

Theorem 4.3. *Let $A \in B(H)^+$ and assume that Tr $(A_-) \leq$ Tr $(A_+) < \infty$ and Tr $(A_+) -$ Tr $(A_-) \in \mathbb{N} \cup \{0\}$. Then A is a strong sum of projections.*

Proof. Since A_+ and A_- are of trace-class and supported in orthogonal subspaces, they are simultaneously diagonalizable, so we can set $A_- = \sum_{i=1}^{M} \lambda_i (f_i \otimes f_i)$, $A_+ = \sum_{j=1}^{N} \mu_j (e_j \otimes e_j)$, where M, $N \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $\{f_i, e_j\}$ are mutually orthogonal unit vectors, and $0 < \lambda_i < 1$, $\mu_j > 0$ or all i and j. Let

$$k := \text{Tr }(A_+) - \text{Tr }(A_-) = \sum_{j=1}^{N} \mu_i - \sum_{i=1}^{M} \lambda_i.$$

Since $\chi_A\{1\}$ is the sum of rank-one projections, we can by (1) assume without loss of generality that

$$A = \sum_{j=1}^{N}(1+\mu_j)(e_j \otimes e_j) + \sum_{i=1}^{M}(1-\lambda_i)(f_i \otimes f_i). \tag{26}$$

By the same proof as in Lemma 2.3 we can decompose A as the sum of k rank-one projections and a positive operator A' with Tr $(A'_+) =$ Tr (A'_-). Thus we assume henceforth that $k = 0$.

We need to consider four cases:
(a) when both A_- and A_+ have finite rank (i.e., $N, M < \infty$),
(b) when A_+ has finite rank and A_- does not (i.e., $N < \infty$, $M = \infty$),
(c) when A_- has finite rank and A_+ does not (i.e., $N = \infty$, $M < \infty$), and
(d) when both have infinite rank (i.e., $N = M = \infty$.)

The case (a) is given by Lemma 2.3 (i).

Consider the case (b). If $N > 1$, choose an $m \in \mathbb{N}$ for which

$$\sum_{j=1}^{N-1}\mu_j < \sum_{i=1}^{m}\lambda_i < \sum_{j=1}^{N}\mu_j, \quad \text{i.e.,} \quad 0 < \sum_{j=1}^{N}\mu_j - \sum_{i=1}^{m}\lambda_i < \mu_N.$$

By Lemma 2.3 (ii) there are $m + N$ rank-one projections P_k for which

$$\sum_{j=1}^{N}(1+\mu_j)(e_j \otimes e_j) + \sum_{i=1}^{m}(1-\lambda_i)(f_i \otimes f_i) = \sum_{k=1}^{m+N-1}P_k + \Big(1 + \sum_{j=1}^{N}\mu_j - \sum_{i=1}^{m}\lambda_i\Big)P_{m+N}$$

Thus

$$A' := A - \sum_{k=1}^{m+N-1}P_k = \Big(1 + \sum_{j=1}^{N}\mu_j - \sum_{i=1}^{m}\lambda_i\Big)P_{m+N} + \sum_{j=m+1}^{\infty}(1-\lambda_j)(f_j \otimes f_j)$$

Since $P_{m+N} \perp f_j$ for all $j > m$ and $\sum_{j=1}^{N}\mu_j - \sum_{i=1}^{m}\lambda_i = \sum_{i=m+1}^{\infty}\lambda_i$, we see that A' satisfies the same conditions as A, but has "$N = 1$". Now we obtain by Lemma 4.1 (ii) that A' is a strong sum of projections and hence so is A.

The next case (c), when $N = \infty$ and $M < \infty$, is similar. If M is not already 1, choose an n for which $\sum_{j=1}^{n-1}\mu_j < \sum_{i=1}^{M-1}\lambda_i < \sum_{j=1}^{n}\mu_j$. Then, again by Lemma 2.3 (ii) there are $M + n - 1$ rank-one projections P_k for which

$$\sum_{j=1}^{n}(1+\mu_j)(e_j \otimes e_j) + \sum_{i=1}^{M-1}(1-\lambda_i)(f_i \otimes f_i) = \sum_{k=1}^{M+n-2}P_k + \Big(1 + \sum_{j=1}^{n}\mu_j - \sum_{i=1}^{M-1}\lambda_i\Big)P_{M+n-1}$$

Set

$$A' := A - \sum_{k=1}^{M+n-2}P_k = (1-\lambda_M)(f_M \otimes f_M) + \Big(1 + \sum_{j=1}^{n}\mu_j - \sum_{i=1}^{M-1}\lambda_i\Big)P_{M+n-1} + \sum_{j=n+1}^{\infty}(1+\mu_j)(e_j \otimes e_j).$$

Since $P_{M+n-1} \leq \sum_{j=1}^{n}(e_j \otimes e_j) + \sum_{i=1}^{M-1}(f_i \otimes f_i)$, P_{M+n-1} is orthogonal to the other rank-one summands of A'. Moreover,

$$\lambda_M = \sum_{j=1}^{n}\mu_j - \sum_{i=1}^{M-1}\lambda_i + \sum_{j=n+1}^{\infty}\mu_j.$$

Now we obtain by Lemma 4.1 (i) that A' is a strong sum of projections and hence so is A.

In the last case (d), both N and M are infinite and $\sum_{j=1}^{\infty}\lambda_j = \sum_{j=1}^{\infty}\mu_j$. Define

$$\Phi_A := \{(m,n)\in\mathbb{N}\times\mathbb{N} \mid \sum_{j=1}^{m}\lambda_j = \sum_{j=1}^{n}\mu_j\}.$$

We need to treat the three possible cases separately, when Φ_A is infinite, when it is finite and non-empty, and when it is empty.

If Φ_A is infinite, then rearrange it as $\Phi_A = \{(m_k, n_k)\}$ where the integer sequences $\{m_k\}$, $\{n_k\}$ are strictly increasing. Set $m_o = n_o = 0$ and by using the unconditional convergence of the series (26), decompose A as

$$A = \sum_{k=1}^{\infty}\Big(\sum_{j=n_{k-1}+1}^{n_k}(1+\mu_j)(e_j\otimes e_j) + \sum_{j=m_{k-1}+1}^{m_k}(1-\lambda_j)(f_j\otimes f_j)\Big).$$

Since $\sum_{j=n_{k-1}+1}^{n_k}\mu_j = \sum_{j=m_{k-1}+1}^{m_k}\lambda_j$, by Corollary 2.5 each summand

$$\sum_{j=n_{k-1}+1}^{n_k}(1+\mu_j)(e_j\otimes e_j) + \sum_{j=m_{k-1}+1}^{m_k}(1-\lambda_j)(f_j\otimes f_j)$$

is a sum of (finitely many) rank-one projections and hence A is strong sum of projections.

If Φ_A is finite but not empty, it has a lexicographically largest element (m,n) for which $\sum_{j=1}^{m}\lambda_j = \sum_{j=1}^{n}\mu_j$ but $\sum_{j=1}^{m'}\lambda_j \neq \sum_{j=1}^{n'}\mu_j$ for any $m' > m$, $n' > n$. Now, again by Corollary 2.5 ,

$$\sum_{j=1}^{n}(1+\mu_j)(e_j\otimes e_j) + \sum_{j=1}^{m}(1-\lambda_j)(f_j\otimes f_j)$$

is the sum of rank-one projections and its remainder

$$A' := A - \sum_{j=1}^{n}(1+\mu_j)(e_j\otimes e_j) + \sum_{j=1}^{m}(1-\lambda_j)(f_j\otimes f_j)$$

satisfies the same conditions as A, but in addition has $\Phi_{A'} = \emptyset$. Finally, the crucial case when $\Phi_A = \emptyset$ is given by Lemma 4.2.

□

In view of the necessary condition established in Theorem 3.3 (i), to conclude our study in $B(H)$ it remains to consider the case when Tr $(A_+) = \infty$. This will be done in Section 6.

5. Type II factors: the finite diagonalizable case

In this section, we assume that M is a type II factor with trace τ. The following key lemma is also a consequence of Lemma 2.1 , or, more precisely, of Lemma 2.6.

Lemma 5.1. *Let $A = (1+\mu)E + (1-\lambda)F$ where E and F are finite projections, $EF = 0$, $\mu \geq 0$, $0 \leq \lambda \leq 1$, and $\tau(A) \geq \tau(R_A)$. Then A is a strong sum of projections.*

Proof. To avoid triviality, assume that $A \neq 0$ and hence $E \neq 0$. The case when $\lambda = 1$ (resp. $\lambda = 0$) is equivalent (resp., implied by) the case when $F = 0$, so assume that $0 < \lambda < 1$, and hence, $R_A = E + F$. If $\mu = 0$, then $-\lambda\tau(F) = \tau(A) - \tau(R_A) \geq 0$, whence $\lambda F = 0$, and then

$A = E + F$ is already a projection. Thus assume henceforth also that $\mu > 0$. Now consider first the key case when $\tau(A) = \tau(R_A)$, i.e., $\mu\tau(E) = \lambda\tau(F)$. In summary, assume that

$$\mu > 0, \quad 0 < \lambda < 1, \quad \text{and} \quad \mu\tau(E) = \lambda\tau(F) > 0. \tag{27}$$

If $\mu = \lambda$, then $\tau(E) = \tau(F)$ and then A is the sum of two (equivalent) projections by Lemma 2.6.

If $\mu < \lambda$, then $\tau(E) > \tau(F)$, and hence, there is some projection $E' \leq E$ with $\tau(E') = \tau(F)$. Then $E' \sim F$ and by Lemma 2.6 there are projections $R_1, F_1 \in M$, $R_1, F_1 \leq E' + F$ with $R_1 \sim F_1 \sim F$ for which

$$(1 + \mu)E' + (1 - \lambda)F = R_1 + (1 + \mu - \lambda)F_1.$$

Set $A_1 := A - R_1$, $E_1 := E - E'$, $\mu_1 := \mu$, and $\lambda_1 := \lambda - \mu$. Then $E_1F_1 = 0$, $E_1 + F_1 \leq E + F$, and

$$\begin{cases} \mu_1 = \mu > 0 \\ 0 < \lambda_1 = \lambda - \mu < 1 \\ \tau(E_1) = \tau(E) - \tau(F) \\ \tau(F_1) = \tau(F) \end{cases}$$

Moreover, $A_1 = (1 + \mu_1)E_1 + (1 - \lambda_1)F_1$ and

$$\mu_1\tau(E_1) = \mu(\tau(E) - \tau(E')) = (\lambda - \mu)\tau(F) = \lambda_1\tau(F_1).$$

Thus A_1 satisfies the same conditions (27) as A does.

Similarly, if $\mu > \lambda$, and hence, $\tau(E) < \tau(F)$, choose a projection $F' \leq F$ with $\tau(F') = \tau(E)$. By the same argument as above, there are projections $R_1, E_1 \in M$, $R_1, E_1 \leq E + F'$, with $R_1 \sim E_1 \sim E$ for which

$$(1 + \mu)E + (1 - \lambda)F' = R_1 + (1 + \mu - \lambda)E_1.$$

Set $F_1 := F - F'$, $\mu_1 := \mu - \lambda$, $\lambda_1 := \lambda$, and $A_1 := A - R_1$. Then, again, $E_1F_1 = 0$, $E_1 + F_1 \leq E + F$, and

$$\begin{cases} \mu_1 = \mu - \lambda > 0 \\ 0 < \lambda_1 = \lambda < 1 \\ \tau(E_1) = \tau(E) \\ \tau(F_1) = \tau(F) - \tau(E) \end{cases}$$

Moreover, $A_1 = (1 + \mu_1)E_1 + (1 - \lambda_1)F_1$ and $\mu_1\tau(E_1) = \lambda_1\tau(F_1)$, i.e., here too A_1 satisfies the conditions (27).

We can thus iterate the construction and find nonzero projections $E_k, F_k, R_k \in M$ with $E_kF_k = 0$, $E_k + F_k \leq E_{k-1} + F_{k-1} \leq E + F$, positive operators $A_k = A_{k-1} - R_k$, and scalars $\mu_k > 0$ and $0 < \lambda_k < 1$ for which $\mu_k\tau(E_k) = \lambda_k\tau(F_k)$, $A_k = (1 + \mu_k)E_k + (1 - \lambda_k)F_k$, and

$$\begin{cases} \mu_{k+1} = \mu_k > 0 \\ 0 < \lambda_{k+1} = \lambda_k - \mu_k < 1 \\ \tau(E_{k+1}) = \tau(E_k) - \tau(F_k) \\ \tau(F_{k+1}) = \tau(F_k) \end{cases} \quad \text{if } \mu_k < \lambda_k \tag{28}$$

$$\begin{cases} \mu_{k+1} = \mu_k - \lambda_k > 0 \\ 0 < \lambda_{k+1} = \lambda_k < 1 \\ \tau(E_{k+1}) = \tau(E_k) \\ \tau(F_{k+1}) = \tau(F_k) - \tau(E_k)x \end{cases} \qquad \text{if } \mu_k > \lambda_k \tag{29}$$

Thus for every k, $A = \sum_{j=1}^{k} R_j + A_k$. This construction terminates if for some k we have $\mu_k = \lambda_k$, in which case A_k is the sum of two projections, and hence, A is the sum of $k+2$ projections. Thus assume henceforth that $\mu_k \neq \lambda_k$ for every k.

By construction, both sequences $\tau(E_k)$ and $\tau(F_k)$ are monotone non-increasing, and hence, both converge. Let $\alpha := \lim \tau(E_k)$ and $\beta := \lim \tau(F_k)$. The sequences μ_k and λ_k are also monotone non-increasing. If $\mu_{k+n} = \mu_k$ for some k and $n \in \mathbb{N}$, then by (28), $\lambda_{k+n} = \lambda_k - n\mu_k$. Thus there must be a largest such n, i.e., the sequence μ_k cannot be eventually constant and, similarly, neither can be the sequence λ_k. Thus, both inequalities $\mu_k < \lambda_k$ and $\mu_k > \lambda_k$ must occur for infinitely many indices. Thus it follows from (28) that $\alpha = \alpha - \beta$, and it follows from (29) that $\beta = \alpha - \beta$, whence $\alpha = \beta = 0$. As a consequence, $||E_k||_1 \to 0$ and $||F_k||_1 \to 0$, and hence, $E_k \underset{s}{\to} 0$ and $F_k \underset{s}{\to} 0$; this implication is well known, the reader is referred to [8, Exercise 8.7.39]. Thus $A_k \underset{s}{\to} 0$, and hence, $A = \sum_{j=1}^{\infty} R_j$ where the convergence is also in the strong operator topology.

We now consider the remaining case when $\mu > 0$, $0 < \lambda < 1$, and $\mu\tau(E) > \lambda\tau(F) \geq 0$. Since M is of type II, we can decompose $E = E_1 + E_2 + E_3$ into the sum of three mutually orthogonal projections with the following traces ($\lfloor \mu \rfloor$ denotes the integer part of μ):

$$\begin{aligned} \tau(E_1) &= \frac{\lambda}{\mu}\tau(F) \geq 0 \\ \tau(E_2) &= \frac{\mu\tau(E) - \lambda\tau(F)}{1 + \lfloor \mu \rfloor} > 0 \\ \tau(E_3) &= \frac{(1 - \mu + \lfloor \mu \rfloor)(\mu\tau(E) - \lambda\tau(F))}{\mu(1 + \lfloor \mu \rfloor)} > 0. \end{aligned}$$

Let

$$\begin{aligned} A_1 &:= (1+\mu)E_1 + (1-\lambda)F \\ A_2 &:= (\mu - \lfloor \mu \rfloor)E_2 + (1+\mu)E_3 = ((1 - (1 + \lfloor \mu \rfloor - \mu)E_2 + (1+\mu)E_3 \\ A_3 &:= (1 + \lfloor \mu \rfloor)E_2 \end{aligned}$$

Thus $A = A_1 + A_2 + A_3$. If $\tau(E_1) = 0$, then $\lambda\tau(F) = 0$, and hence, $A_1 = F$ is already a projection. If $\tau(E_1) \neq 0$, then A_1 satisfies the conditions of (27), and hence, it is a strong sum of projections. If $\mu \in \mathbb{N}$, then $A_2 = (1+\mu)E_3$ is the sum of $1+\mu$ projections. If $\mu \neq \lfloor \mu \rfloor$, then it is easy to verify that also A_2 satisfies the conditions of (27), and hence, is a strong sum of projections. Finally, A_3 is always trivially the sum of $1 + \lfloor \mu \rfloor$ projections, which concludes the proof.

□

Now we consider positive diagonalizable operators in M, namely, those operators of the form $A = \sum_k \gamma_k G_k$ where $G_k \in M$ are mutually orthogonal projections and $\gamma_k > 0$, and the series, if infinite, converges in the strong operator topology.

Theorem 5.2. *Let M be a type II factor with trace τ and let $A \in M$ be a positive diagonalizable operator. If $\tau(A_+) \geq \tau(A_-)$, then A is a strong sum of projections.*

Proof. To avoid triviality, assume that $A \neq 0$. By renaming appropriately the coefficients and using the semifiniteness of M to split the projections into sums of projections with finite trace, we rewrite A as

$$A = \sum_{j=1}^{N} (1 + \mu_j) E_j + \sum_{i=1}^{K} (1 - \lambda_i) F_i$$

with $N, K \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, E_j, F_i mutually orthogonal finite projections, $\mu_j > 0$, and $0 \leq \lambda_i < 1$ for all j and i, and with the series converging strongly if N or K are infinite. Again, we use the convention that if N or K are zero then A is the sum of only one series. Since $\sum\{(1 - \lambda_i)F_i \mid \lambda_i = 0\}$ is already a projection, we can further assume without loss of generality that $\lambda_i > 0$ for all i. Then

$$A_+ = \sum_{j=1}^{N} \mu_j E_j, \quad A_- = \sum_{i=1}^{K} \lambda_i F_i, \quad \text{and hence,} \quad \sum_{j=1}^{N} \mu_j \tau(E_j) \geq \sum_{j=1}^{K} \lambda_j \tau(F_j).$$

In particular, $N > 0$. Assume $K > 0$. Then $\lambda_1 \tau(F_1) \leq \sum_{j=1}^{N} \mu_j \tau(E_j)$. Since M is of type II, we can find projections $E_{j1} \leq E_j$ such that $\lambda_1 \tau(F_1) = \sum_{j=1}^{N} \mu_j \tau(E_{j1})$. Then decompose $F_1 = \sum_{j=1}^{N} F_{j1}$ into mutually orthogonal projections so that $\lambda_1 \tau(F_{j1}) = \mu_j \tau(E_{j1})$ for every j. If $K > 1$ we have

$$\sum_{i=2}^{K} \lambda_i \tau(F_i) \leq \sum_{j=1}^{N} \mu_j \tau(E_j - E_{j1}),$$

and hence, we can iterate the process. Thus for every i and j we decompose $F_i = \sum_{j=1}^{N} F_{ji}$ into mutually orthogonal projections and further find mutually orthogonal projections $E_{ji} \leq E_j$ so that $\lambda_i \tau(F_{ji}) = \mu_j \tau(E_{ji})$. Set $E_{jo} := E_j - \sum_{i=1}^{K} E_{ji}$. Then

$$A = \sum_{j=1}^{N} \sum_{i=1}^{K} \Big((1 + \mu_j) E_{ji} + (1 - \lambda_i) F_{ji} \Big) + \sum_{j=1}^{N} (1 + \mu_j) E_{jo}.$$

By Lemma 5.1, each summand $(1 + \mu_j)E_{ji} + (1 - \lambda_i)F_{ji}$ and $(1 + \mu_j)E_{jo}$ is a strong sum of projections, and hence, so is A. In the case that $K = 0$, $A = \sum_{j=1}^{N} (1 + \mu_j) E_j$, and hence, it is also the strong sum of projections by the same reasoning. □

As the following examples show, the condition that A is diagonalizable is not necessary for A to be a strong sum of projections.

Example 5.3. *Let M be a type II_1 factor, let $P \in M$ be a projections with $P \sim P^\perp$, let $\mathcal{A}$ (resp., $\mathcal{B}$) be a masa in M_P (resp., in $M_{P^\perp}$). By properly scaling the spectral resolution of a generator of $\mathcal{A}$ we can find a monotone increasing strongly continuous net of projections $\{E_t\}_{t \in [0, \frac{1}{2}]}$ in $\mathcal{A}$ with $\tau(E_t) = t$.*

(i) *Assume that $\mathcal{A}$ and $\mathcal{B}$ are conjugate in M, and hence there is a selfadjoint unitary $U \in M$ for which $U\mathcal{A}U^* = \mathcal{B}$. Define*

$$E_t := I - U E_{1-t} U^* \text{ for } t \in (\tfrac{1}{2}, 1] \quad \text{and} \quad A = \int_0^{\frac{1}{2}} (1 + t) dE_t + \int_{\frac{1}{2}}^{1} t dE_t.$$

Then $\{E_t\}_{t\in[0,1]}$ is flag, namely a monotone increasing strongly continuous net of projections with $\tau(E_t)=t$ for all $t\in[0,1]$ and A is not diagonalizable (in fact, it has no eigenvalues). Furthermore, $0\le A\le \frac{3}{2}I\le 2I$, $R_A=I$, and it is easy to verify that

$$\begin{aligned}UAU^* &= \int_0^{\frac{1}{2}}(1+t)d(UE_tU^*)+\int_{\frac{1}{2}}^1 td(UE_tU^*)\\ &= \int_0^{\frac{1}{2}}(1+t)d(I-E_{1-t})+\int_{\frac{1}{2}}^1 td(I-E_{1-t})\\ &= \int_{\frac{1}{2}}^1(2-t)dE_t+\int_0^{\frac{1}{2}}(2-(1+t))dE_t\\ &= 2I-A.\end{aligned}$$

Thus by Proposition 2.1 , A is the sum of two projections.

(ii) *Assume that $\mathcal{A}$ and $\mathcal{B}$ are not conjugate in M. Such a case can be easily obtained by choosing P so that $M\sim M_P\sim M_{P^\perp}$, choosing two non-conjugate masas $\mathcal{A}_o$ and $\mathcal{B}_o$ in M (e.g., a Cartan masa and a singular one) and defining $\mathcal{A}$ and $\mathcal{B}$ to be the compressions of $\mathcal{A}_o$ and $\mathcal{B}_o$ to M_P and $M_{P^\perp}$ respectively. Complete $\{E_t\}_{t\in[0,\frac{1}{2}]}$ to be a flag in M by defining $E_t:=P+F_t$ for $t\in(\frac{1}{2},1]$ where $\{F_t\}_{t\in[\frac{1}{2},1]}$ is an arbitrary monotone increasing strongly continuous net of projections in $\mathcal{B}$ with $\tau(F_t)=t-\frac{1}{2}$. Define as in (i)*

$$A=\int_0^{\frac{1}{2}}(1+t)dE_t+\int_{\frac{1}{2}}^1 tdE_t.$$

Again, A is not diagonalizable, in fact it has no, $0\le A\le\frac{3}{2}I\le 2I$, and $R_A=I$. By Proposition 2.10 , A is the sum of two projections if and only if $UAU^*=2I-A$ for some unitary $U\in M$. Reasoning as in the proof of Proposition 2.10 , it is simple to see that if such a unitary existed, we would have $\mathcal{B}=U\mathcal{A}U^*$ and hence $\mathcal{A}$ and $\mathcal{B}$ would be conjugate, against the assumption. Thus A cannot be the sum of two projections in M. However we do not know whether A is a strong sum of projections in M or not.

Question 5.4. *Can the condition that A is diagonalizable be removed from Theorem 5.2?*

6. The infinite case

In this section we assume that M is an infinite factor, i.e., of type I_∞, type II_∞, or type III. The following lemma is the key to the proof of Theorem 1.1 in this case.

Lemma 6.1. *Let $A=\sum_{j=1}^\infty(1+\mu_j)E_j+(1-\lambda)F$ where $\{E_j,F\}$ are mutually orthogonal equivalent projections in M, $\mu_j>0$, $0<\lambda_j\le 1$, and $\sup\mu_j<\infty$. If $\sum_{j=1}^\infty\mu_j=\infty$, then A is a strong sum of projections in M.*

Proof. Let $n_1\ge 1$ be the smallest integer for which $\sum_{j=1}^{n_1}\mu_j\ge\lambda$. Such an integer exists because $\sum_{j=1}^\infty\mu_j=\infty$. Set

$$\alpha_1:=\begin{cases}\lambda & \text{if } n_1=1\\ \lambda-\sum_{j=1}^{n_1-1}\mu_j & \text{if } n_1>1\end{cases}\quad\text{and}\quad 1-\beta_1:=\mu_{n_1}-\alpha_1-\lfloor\mu_{n_1}-\alpha_1\rfloor$$

where $\lfloor x \rfloor$ denotes the integer part of x. Then $\mu_{n_1} \geq \alpha_1$, $0 < \alpha_1 \leq 1$, and $0 < \beta_1 \leq 1$. The positive operator

$$D_1 := \sum_{j=1}^{n_1-1}(1+\mu_j)E_j + (1+\alpha_1 + \lfloor \mu_{n_1} - \alpha_1 \rfloor)E_{n_1} + (1-\lambda)F$$

is a linear combination of $n := \begin{cases} n_1+1 & \text{if } \lambda \neq 1 \\ n_1 & \text{if } \lambda = 1 \end{cases}$ mutually orthogonal equivalent projections in M and the sum of their coefficients is $k_1 := n_1 + 1 + \lfloor \mu_{n_1} - \alpha_1 \rfloor$. Since $k_1 \in \mathbb{N}$ and $k_1 \geq n$, by Lemma 2.6 (ii), D_1 is the sum of k_1 (equivalent) projections.

Next, we apply the same construction to the "remainder"

$$A - D_1 = \sum_{j=n_1+1}^{\infty}(1+\mu_j)E_j + (1-\beta_1)E_{n_1}$$

where now β_1 plays the role of λ and E_{n_1} the role of F. Iterating we find an increasing sequence of indices n_k and two sequences of positive numbers $0 < \alpha_k, \beta_k \leq 1$ with $\mu_{n_k} \geq \alpha_k$ and $1 - \beta_k = \mu_{n_k} - \alpha_k - \lfloor \mu_{n_k} - \alpha_k \rfloor$. Then the positive operator

$$D_k := \sum_{j=n_{k-1}+1}^{n_k-1}(1+\mu_j)E_j + (1+\alpha_k + \lfloor \mu_{n_k} - \alpha_k \rfloor)E_{n_k} + (1-\beta_{k-1})E_{n_{k-1}}$$

is by Lemma 2.6 the sum of finitely many (equivalent) projections. But then

$$A - \sum_{j=1}^{k} D_j = \sum_{j=n_k+1}^{\infty}(1+\mu_j)E_j + (1-\beta_k)E_{n_k} \underset{s}{\to} 0 \tag{30}$$

because the projections $\{E_j\}$ are mutually orthogonal. Thus $A = \sum_{j=1}^{\infty} D_j$, where the series converges in the strong operator topology. Since each D_k is the sum of projections, so is A. $\square$

If M is of type I and all projections E_j and F have rank-one, then we can relax the condition that they are mutually orthogonal. Indeed, orthogonality is not necessary to conclude that each positive finite rank operator D_k is the sum of projections (see Corollary 2.6 and also Lemma 2.2), and assuming strong convergence of the series $\sum_{j=1}^{\infty}(1+\mu_j)E_j$ is sufficient to guarantee that $\sum_{j=n_k+1}^{\infty}(1+\mu_j)E_j + (1-\beta_k)E_{n_k} \underset{s}{\to} 0$ in (30). Thus we have:

Lemma 6.2. *Let $A = \sum_{j=1}^{\infty}(1+\mu_j)E_j + (1-\lambda)F$ where E_j, $F \in B(H)$ are rank-one projections, $\mu_j > 0$, $0 < \lambda_j \leq 1$, and $\sum_{j=1}^{\infty}(1+\mu_j)E_j$ converges in the strong operator topology. If $\sum_{j=1}^{\infty}\mu_j = \infty$, then A is a strong sum of projections.*

Proposition 6.3. *Let $A \in M^+$ and assume that there is some $\mu > 0$ for which the spectral projection $\chi_A[1+\mu, \infty)$ is infinite. Then A is a strong sum of projections.*

Proof. Let $E := \chi_A[1+\mu, \infty)$, $B := A - (1+\mu)E$, and let $\mathcal{A}$ be a masa containing A. Then $B \in \mathcal{A}$. By [18, Corollary 2.23], B can be decomposed into a norm converging series $B = \sum_{i=1}^{\infty}(1-\lambda_i)Q_i$ with $0 \leq \lambda_i < 1$ and with the projections $Q_i \in \mathcal{A}$. (In fact we can choose $1 - \lambda_i = 2^{-i}$, but we do not need this fact here.) Some or all of the projections Q_i can be zero. Since M is infinite and $E \in \mathcal{A}$, by [7, Theorem 3.18] (see also [11, Corollary 31]), we can decompose $E = \sum_{i=1}^{\infty} E_i$ into a sum of infinite projections $E_i \in \mathcal{A}$. Let

$A_i := (1+\mu)E_i + (1-\lambda_i)Q_i$. Then $A = \sum_{i=1}^{\infty} A_i$. Thus it suffices to prove that A_i is a strong sum of projections for each i. Using the fact that $E_i,\ Q_i \in \mathcal{A}$, and hence, they commute, it follows that A_i is diagonalizable as

$$A_i = (1+\mu)(E_i - E_iQ_i) + (2+\mu-\lambda_i)E_iQ_i + (1-\lambda_i)(Q_i - E_iQ_i).$$

Since E_i is infinite, at least one of the two orthogonal projections $E_i - E_iQ_i$ and E_iQ_i must be infinite. Assume that $E_i - E_iQ_i$ is infinite. If $Q_i = 0$, then $A_i = (1+\mu)E_i$ and the conclusion follows from Lemma 6.1 by further decomposing E_i into a sum of infinitely many mutually orthogonal equivalent projections.

Thus assume that $Q_i \neq 0$ and decompose $2+\mu-\lambda_i = \sum_{n=1}^{m}(1-\gamma_n)$ into the sum of finitely many numbers $0 < 1-\gamma_n < 1$. Next, decompose $E_i - E_iQ_i = \sum_{n=1}^{m+1} E_i^{(n)}$ into the sum of $m+1$ mutually orthogonal equivalent (infinite) projections $E_i^{(n)}$. Then further decompose each projection $E_i^{(n)}$ into a sum of infinitely many mutually orthogonal projections $E_{ij}^{(n)}$ with

$$E_{ij}^{(n)} \sim \begin{cases} E_iQ_i & \text{for } 1 \leq n \leq m \\ Q_i - E_iQ_i & \text{for } n = m+1. \end{cases}$$

Thus $E_i - E_iQ_i = \sum_{n=1}^{m+1}\sum_{j=1}^{\infty} E_{ij}^{(n)}$. Define

$$B_i^{(n)} = \begin{cases} \sum_{j=1}^{\infty}(1+\mu)E_{ij}^{(n)} + (1-\gamma_n)E_iQ_i & \text{for } 1 \leq n \leq m \\ \sum_{j=1}^{\infty}(1+\mu)E_{ij}^{(m+1)} + (1-\lambda_i)(Q_i - E_iQ_i) & \text{for } n = m+1. \end{cases}$$

By construction, $A_i = \sum_{n=1}^{m+1} B_i^{(n)}$. By Lemma 6.1 , all the operators $B_i^{(n)}$ are strong sums of projections and hence so is A_i. Finally, the case when E_iQ_i is infinite is similar and is left to the reader.

□

An immediate consequence of this proposition is the sufficient condition in Theorem 1.1 (iii) for the type III case.

Corollary 6.4. *Let M be a type III factor, $A \in M^+$, and either A be a projection or A satisfy $||A|| > 1$. Then A is a strong sum of projections.*

Proof. If $||A|| > 1$, then there is some $\mu > 0$ for which the spectral projection $\chi_A[1+\mu, \infty)$ is nonzero and hence infinite. Then A is a strong sum of projections by Proposition 6.3.

□

Remark 6.5.

(i) *The condition that $\chi_A[1+\mu,\infty)$ is infinite for some $\mu > 0$ is equivalent to the condition $||A||_{ess} > 1$ where $||A||_{ess}$ is the essential norm, i.e., the norm in the quotient M/K, where K is the norm closed ideal generated by the finite projections of M.*

If $M = B(H)$, then K is the ideal of compact operators $K(H)$ on H and Proposition 6.3 provides another proof of [5, Theorem 2] *stating that if $||A||_{ess} > 1$, then A is a strong sum of projections.*

If M is of type II_∞, K is the ideal of compact operators relative to M introduced by Sonis [17] *and Breuer* [4] *(see also* [10]*).*

If M is of type III, then $K = \{0\}$ and $||A||_{ess} = ||A||$.

(ii) *If M is semifinite and $A \in K^+$ is a strong sum of projections then $\tau(R_A) < \infty$.*

Proof.

(ii) It is well known that $\tau(\chi_A(\gamma,\infty)) < \infty$ for every $\gamma > 0$ (e.g., see [10, Theorem 1.3]). In particular, $\tau(\chi_A(1,\infty)) < \infty$, whence $\tau(A_+) < \infty$. Thus it follows from Theorem 3.3 that $\tau(A_-) < \infty$. But $A_- \geq \frac{1}{2}\chi_A(0,\frac{1}{2}]$, hence $\tau(\chi_A(0,\frac{1}{2})) < \infty$. From this follows that $\tau(\chi_A(0,\infty)) = \tau(\chi_A(0,\frac{1}{2}]) + \tau(\chi_A(\frac{1}{2},\infty)) < \infty$ □

An alternative proof of (ii) for the case when $M = B(H)$ is that a strongly converging series of rank-one projections that converges to a compact operator must converge uniformly and hence be finite.

We can now prove the last part of the sufficiency in Theorem 1.1.

Theorem 6.6. *Let M be type I_∞ or type II_∞. If $\tau(A_+) = \infty$. Then A is a strong sum of projections.*

Proof. By Proposition 6.3, we just need to consider the case when $\chi_A[1+\mu,\infty)$ is finite for every $\mu > 0$. Let $\mathcal{A}$ be a masa of M containing A. Let $E'_1 := \chi_A[2,\infty) = \chi_A[2,||A||]$ and $E'_j := \chi_A[1+\frac{1}{j}, 1+\frac{1}{j-1})$ for $j > 1$. Then $\tau(E'_j) < \infty$ for all j. Since

$$\sum_{j=1}^{\infty} \frac{1}{j} E'_j \leq A_+ \leq (||A||-1)E'_1 + \sum_{j=2}^{\infty} \frac{1}{j-1} E'_j,$$

we see that $(||A||-1)\tau(E'_1) + \sum_{j=2}^{\infty} \frac{1}{j-1}\tau(E'_j) = \infty$. Then also $\sum_{j=1}^{\infty} \frac{1}{j}\tau(E'_j) = \infty$. Furthermore, $A\chi_A(1,||A||] - \sum_{j=1}^{\infty}(1+\frac{1}{j})E'_j \in \mathcal{A}^+$, $A\chi_A[0,1] \in \mathcal{A}^+$, and

$$A = \sum_{j=1}^{\infty}(1+\frac{1}{j})E'_j + A\chi_A[0,1] + \Big(A\chi_A(1,||A||] - \sum_{j=1}^{\infty}(1+\frac{1}{j})E'_j\Big).$$

Now we consider separately the case when $M = B(H)$ and when M is of type II.

If $M = B(H)$, first decompose by [18, Corollary 2.23] the positive operator

$$B := A\chi_A[0,1] + \Big(A\chi_A(1,||A||] - \sum_{j=1}^{\infty}(1+\frac{1}{j})E'_j\Big)$$

into a norm converging series $B = \sum_{i=1}^{\infty}(1-\lambda'_i)Q'_i$ with $0 < \lambda'_i \leq 1$ and with the projections $Q'_i \in \mathcal{A}$. Some or all of the projections Q'_i can be zero. Then, further decompose the projections E'_j and Q'_i into rank-one projections. Relabel the ensuing sequence of coefficients $1+\frac{1}{j}$ (resp., $1-\lambda'_i$) repeated according to the multiplicity of the projections as $1+\mu_j$ (resp. $1-\lambda_i$). To take into account the case when there are only finitely many non-zero projections Q'_i and they all have finite rank, allow $\lambda_i = 1$. Thus

$$A = \sum_{j=1}^{\infty}(1+\mu_j)E_j + \sum_{i=1}^{\infty}(1-\lambda_i)Q_i$$

where all the projections E_j and Q_i have rank-one, $\mu_j > 0,\ 0 < \lambda_i \leq 1$ for all i and j, both series converge in the strong operator topology, and $\sum_{j=1}^{\infty}\mu_j = \infty$. Now further decompose $\mathbb{N} = \cup_{i=1}^{\infty}\Lambda_i$ into infinite disjoint subsets Λ_i so that for each i, $\sum_{j\in\Lambda_i}\mu_j = \infty$. Then

$$A = \sum_{i=1}^{\infty}\left(\sum_{j\in\Lambda_i}(1+\mu_j)E_j + (1-\lambda_i)Q_i\right)$$

and each summand being the strong sum of projection by Lemma 6.2, so is A.

Now assume that M is of type II and again using [18, Corollary 2.23] decompose separately $A\chi_A[0,1]$ and $A\chi_A(1, ||A||]-\sum_{j=1}^{\infty}(1+\frac{1}{j})E_j$ into two norm converging series of scalar multiples $1-\lambda_i'$ of projections $Q_i' \in \mathcal{A}$. If a projection Q_i' in the series decomposing $A\chi_A[0,1]$ is infinite, by the semifiniteness of M we can further decompose it into a strongly converging sum of mutually orthogonal finite projections in M. These projections are not necessarily in $\mathcal{A}$, however, being majorized by $\chi_A[0,1]$, they are all orthogonal to and hence commute with all the projections E_j'.

Every projection Q_i' in the series decomposing $A\chi_A(1, ||A||]-\sum_{j=1}^{\infty}(1+\frac{1}{j})E_j$ is in $\mathcal{A}$, is majorized by $\chi_A(1, ||A||] = \sum_{j=1}^{\infty} E_j$, and hence is the sum $Q_i' = \sum_{j=1}^{\infty} Q_i'E_j$ of finite projections $Q_i'E_j'$ which belong to $\mathcal{A}$ and hence commute with all the projections E_j'. Therefore,

$$A = \sum_{j=1}^{\infty}(1+\frac{1}{j})E_j' + \sum_{i=1}^{\infty}(1-\lambda_i)Q_i$$

where for all i, $0 \leq \lambda_i < 1$ and Q_i are finite projections that commute with each E_j. Since $\sum_{j=1}^{\infty}\frac{1}{j}\tau(E_j') = \infty$, we can choose an increasing sequence of indices n_i for which $\sum_{j=n_i+1}^{n_{i+1}}\frac{1}{j}\tau(E_j') \geq \lambda_i\tau(Q_i)$ and let $A_i := \sum_{j=n_i+1}^{n_{i+1}}(1+\frac{1}{j})E_j' + (1-\lambda_i)Q_i$. Then $A = \sum_{i=1}^{\infty} A_i$ and

$$\tau(A_i) = \sum_{j=n_i+1}^{n_{i+1}}\tau(E_j') + \tau(Q_i) + \sum_{j=n_i+1}^{n_{i+1}}\frac{1}{j}\tau(E_j') - \lambda_i\tau(Q_i) \geq \tau\big(\big(\sum_{j=n_i+1}^{n_{i+1}}E_j'\big)\vee Q_i\big) = \tau(R_{A_i})$$

Since A_i is diagonalizable because Q_i commutes with all E_j', A_i is a strong sum of projections by Theorem 5.2 and hence so is A. □

References

[1] Antezana, J., Massey, P., Ruiz, M, and Stojanoff, D., *The Schur-Horn Theorem for operators and frames with prescribed norms and frame operator,* Illinois J of Math. to appear.

[2] Argerami, M., and Massey, P. *A Schur-Horn theorem in type II_1 factors* Preprint

[3] Argerami, M., and Massey, P. *Towards the carpenter's theorem* Preprint.

[4] Breuer, M. *Fredholm theories in von Neumann algebras. I and II* Math. Ann. **178** (1968), 243-254 and Math. Ann. **180** (19689), 313-325.

[5] Dykema, K., Freeman, D., Kornelson, K., Larson, D., Ordower, M., and Weber, E., *Ellipsoidal tight frames and projection decompositions of operators* Illinois J of Math. **48**, (2004), 477-489.

[6] Fillmore, P., *On sums of projections* J. Func. Anal., **4** (1969), 146-152.

[7] Kadison, R.V., *Diagonalizing matrices,* Amer. J. Math. **106**, (1984), 1451-1468

[8] Kadison, R.V. and Ringrose, J. *Fundamentals of the Theory of Operator Algebras II* Pure and Appl Math, London, N.Y., Paris, Academic press, 1985.

[9] Kadison, R., *The Pythagorean Theorem II: the infinite discrete case,* Proc. Natl. Acad. Sci. USA **99** 8 (2002), 5217–5222.

[10] Kaftal, V., *On the theory of compact operators in von Neumann algebras I,* Indiana Univ. Math. J. **26**, (1977), 447-457.

[11] Kaftal, V., *Type decomposition for von Neumann algebra embeddings* J. Func. Anal **98**, No. 1, (1991), 169-193.

[12] Kaftal, V., Ng, P. W., and Zhang. S., *Projection decomposition in multiplier algebras* Preprint

[13] Kaftal, V., and Weiss, G., *An infinite dimensional Schur-Horn theorem and majorization theory with applications to operator ideals* Preprint

[14] Kaftal, V., and Larson, D., *Orthogonality free finite frames and perturbations of projections* Preprint

[15] Kaftal, V., and Larson, D., *Admissible sequences for positive operators* Preprint

[16] Kornelson, K. and Larson, D., *Rank-one decompositions of operators and construction of frames,* Wavelets, frames and operator theory, Contemp. Math **345**, Amer. Math. Soc. Providence, Ri (2004), 203–214.
[17] Sonis, M. G., *On a class of operators in von Neumann algebras with Segal measures,* Math USSR Sbornik, **13** (1971), 3, 344-359.
[18] Stratila, S. and Zsido, L. *Lectures on von Neumann Algebras.* Abacus Press, Bucharest, 1979.
[19] Takesaki, M. *Theory of Operator Algebras I.* Springer-Verlag, New York Heidelberg Berlin, 1979.

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